Published for SISSA by Springer

Received: November 3, 2017
Accepted: December 8, 2017
Published: December 19, 2017

T-folds from Yang-Baxter deformations

José J. Fernández-Melgarejo,[a,b] **Jun-ichi Sakamoto,**[c] **Yuho Sakatani**[d,e] **and Kentaroh Yoshida**[c]

[a] *Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto, Japan*

[b] *Departamento de Física, Universidad de Murcia, Murcia, Spain*

[c] *Department of Physics, Kyoto University, Kyoto, Japan*

[d] *Department of Physics, Kyoto Prefectural University of Medicine, Kyoto, Japan*

[e] *Fields, Gravity & Strings, CTPU, Institute for Basic Sciences, Daejeon, South Korea*

E-mail: josejuan@yukawa.kyoto-u.ac.jp, sakajun@gauge.scphys.kyoto-u.ac.jp, yuho@koto.kpu-m.ac.jp, kyoshida@gauge.scphys.kyoto-u.ac.jp

Abstract: Yang-Baxter (YB) deformations of type IIB string theory have been well studied from the viewpoint of classical integrability. Most of the works, however, are focused upon the local structure of the deformed geometries and the global structure still remains unclear. In this work, we reveal a non-geometric aspect of YB-deformed backgrounds as T-fold by explicitly showing the associated $\mathrm{O}(D,D;\mathbb{Z})$ T-duality monodromy. In particular, the appearance of an extra vector field in the generalized supergravity equations (GSE) leads to the non-geometric Q-flux. In addition, we study a particular solution of GSE that is obtained by a non-Abelian T-duality but cannot be expressed as a homogeneous YB deformation, and show that it can also be regarded as a T-fold. This result indicates that solutions of GSE should be non-geometric quite in general beyond the YB deformation.

Keywords: String Duality, Supergravity Models, Integrable Field Theories, AdS-CFT Correspondence

ArXiv ePrint: 1710.06849

https://doi.org/10.1007/JHEP12(2017)108

JHEP12(2017)108

Contents

1 Introduction

A prototypical example of the AdS/CFT correspondence [1] is a conjectured equivalence between a type IIB superstring theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$ and the four-dimensional $\mathcal{N} = 4$ $\mathrm{SU}(N)$ super Yang-Mills theory in the large N limit. Nowadays, it is well recognized that an integrable structure underlies this correspondence (for a comprehensive review, see [2]). In particular, the associated supergeometry is represented by a supercoset [3],

$$\frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4) \times \mathrm{SO}(5)}\,, \tag{1.1}$$

and the $\mathbb{Z}_4$-grading of it ensures the classical integrability of the supercoset sigma model [4].

A renewed interest for this integrable structure appeared from the development of a systematic scheme of integrable deformation called the Yang-Baxter (YB) deformation [5–9]. In fact, an application of this scheme to type IIB superstring on $\mathrm{AdS}_5\times\mathrm{S}^5$ [10–12] opened up a lot of new perspectives and directions including intriguing relations among YB deformations and non-commutative gauge theories, non-Abelian T-duality [13–22], and manifestly T-/U-duality covariant formulations, which have been discovered in [23–29], and [30–33], respectively. Our concern here is to delve deeper into the relation to a manifestly T-duality covariant formulation called Double Field Theory (DFT) [34–38] with particular emphasis on non-geometric aspects (i.e., the global nature) of YB-deformed backgrounds.

The original application of YB deformations to string theory is the standard q-deformation of type IIB string on $\mathrm{AdS}_5\times\mathrm{S}^5$ [10, 11] with a classical r-matrix of Drinfeld-Jimbo type [39, 40]. The metric and Neveu-Schwarz-Neveu-Schwarz (NS-NS) 2-form of the deformed background were computed in [41] by performing coset construction for the bosonic group elements. Note here that the deformed background is called in several ways, as the q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$, the η-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$, or the ABF background, but all of them are identical.[1] The supercoset construction for the q-deformed case was worked out in [42]. While the q-deformation is based on the modified classical Yang-Baxter equation (mCYBE), one may consider another category based on the homogeneous CYBE [9, 12], for which a large number of works [23–25, 29, 43–56] have been done and led to various backgrounds including well-known examples such as Lunin-Maldacena-Frolov backgrounds [57, 58], gravity duals of non-commutative gauge theories [59, 60] and Schrödinger spacetimes [61–63].

Remarkably, the full q-deformed background of [42], which includes the Ramond-Ramond (R-R) fluxes and dilaton, is not a solution of type IIB supergravity. Afterward, it was shown that the background should satisfy the generalized supergravity equations of motion (GSE) [64]. At that moment, the GSE seemed to be an artifice invented so as to support the q-deformed background as a solution. However, after that, in the ground-breaking paper by Tseytlin and Wulff [65], this GSE has been reproduced by solving the kappa-symmetry constraints of the Green-Schwarz type IIB string theory on an arbitrary background. Therefore, the GSE has now been established on the fairly fundamental ground.

For the homogeneous CYBE case [12], there exists a significant criterion to identify whether a YB-deformed background is a solution of type IIB supergravity or GSE, before deriving the concrete expression of the resulting deformed background, namely, at the level of classical r-matrix. It is called the unimodularity condition [55]. When this condition is satisfied, the deformed background is a solution of type IIB supergravity, but if not, the background is a solution of the GSE. Various solutions of GSE have been obtained from YB deformations with non-unimodular classical r-matrices [52, 54]. As it has been shown in [23–26], classical r-matrices, which characterize homogeneous YB deformations of AdS_5 geometry, are closely related to non-commutative parameters in the dual open-string description and, as pointed out in [33], they are nothing but β-fields [66]. In terms of the

[1] It should be remarked that the η-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$ means all of the YB deformations of $\mathrm{AdS}_5\times\mathrm{S}^5$ in some literature, but here we will not follow that convention.

β-field, the non-unimodularity is measured as [25, 26, 33]

$$\frac{1}{\sqrt{|G|}}\,\partial_m(\sqrt{|G|}\,\beta^{nm}) \neq 0\,, \tag{1.2}$$

where G_{mn} is the so-called the open-string metric that will be defined later. The quantity on the left-hand side is basically the trace of a non-geometric Q-flux, and this result indicates that YB deformations with non-unimodular r-matrices lead to non-geometric backgrounds.

In this paper, we will concentrate on YB deformations of Minkowski and $\mathrm{AdS}_5 \times \mathrm{S}^5$ backgrounds, and find that the deformed backgrounds we consider here belong to a specific class of non-geometric backgrounds, called T-folds [67]. As far as we know, the YB-deformed backgrounds have not been recognized as T-folds so far, hence this is the first work that clearly states the relation between YB-deformed backgrounds and T-folds. Moreover, it is worth noting that our examples have an intriguing feature that the R-R fields are also twisted by the T-duality monodromy, in comparison to the well-known T-folds which include no R-R fields.

This paper is organized as follows. Section 2 provides a brief review of T-folds, including two examples that are well-known in the literature. In section 3, we consider GSE solutions which can be realized as YB deformations of Minkowski spacetime and $\mathrm{AdS}_5\times\mathrm{S}^5$, and argue that these deformed backgrounds are regarded as T-folds. In addition, we study a solution of GSE that is obtained by a non-Abelian T-duality but not as a YB deformation, and show that this can also be regarded as a T-fold. Section 4 is devoted to conclusions and discussions. In appendix A, we discuss how to generalize the Penrose limit [68, 69] so as to produce various GSE solutions. To be pedagogical, appendix A.1 is devoted to a review of Penrose limit of Poincaré AdS_5. In appendix A.2, we discuss the modified Penrose limit with a rescaling of the deformation parameter. Then, we apply it to YB-deformed background and reproduce the deformed Minkowski backgrounds discussed in section 3.4.

2 A brief review of T-folds

In this section, let us explain what is T-fold. A T-fold is supposed to be a generalization of the usual manifold. It locally looks like a Riemannian manifold, but which is glued together not just by diffeomorphisms but also by T-duality. It plays a significant role in studying non-geometric fluxes beyond the effective supergravity description. As illustrative examples, we revisit two well-known cases in the literature, corresponding to a chain of duality transformations [70, 71] and to the codimension-1 5^2_2-brane solution [72].

It is conjectured that string theories are related by some discrete dualities. One thing that can occur is that, by duality transformations, a flux configuration transforms into a non-geometric flux configuration, which means that it cannot be realized in terms of the usual fields in 10/11-dimensional supergravities. Therefore, dualities suggest that we need to go beyond the usual geometric isometries to fully understand the arena of flux compactifications.

For the case of T-duality, one proposal to address this problem is the so-called doubled formalism. This construction consists of a manifold in which all the local patches are geometric. However, the transition functions that are needed to glue these patches not only include usual diffeomorphisms and gauge transformations, but also T-duality transformations.

T-fold backgrounds are formulated in an enlarged space with a $T^n \times \tilde{T}^n$ fibration. The tangent space is the doubled torus $T^n \times \tilde{T}^n$ and is described by a set of coordinates $Y^M = (y^m, y_m)$ which transforms in the fundamental representation of $\mathrm{O}(n,n)$. The physical internal space arises as a particular choice of a subspace of the double torus, $T^n_{\text{phys}} \subset T^n \times \tilde{T}^n$. Then T-duality transformations $\mathrm{O}(n,n;\mathbb{Z})$ act by changing the physical subspace T^n_{phys} to a different subspace of the enlarged $T^n \times \tilde{T}^n$. For a geometric background, we have a spacetime which is a geometric bundle, $T^n_{\text{phys}} = T^n$.[2] Nevertheless non-geometric backgrounds do not fit together to form a conventional manifold. That is to say, despite of they are locally well-defined, their global description is not valid. Instead, they are globally well-defined as T-folds.

This formulation is manifestly invariant under the T-duality group $O(n,n;\mathbb{Z})$. However, to make contact with the conventional formulation, one needs to choose a polarization, i.e., a particular choice of $T^n_{\text{phys}} \subset T^n \times \tilde{T}^n$. This means that we have to break the $\mathrm{O}(n,n;\mathbb{Z})$ and pick n coordinates out of the $2n$ coordinates $(y^m, \tilde{y}_m)$. Then, T-duality transformations allow to identify the backgrounds that belong to the same physical configuration or duality orbit and just differ on a choice of polarization.[3]

Due to the $\mathrm{O}(n,n)$ symmetry, it is convenient to introduce the generalized metric $\mathcal{H}_{MN}$ on the double torus,

$$
\begin{aligned}
&\mathcal{H} \equiv (\mathcal{H}_{MN}) \equiv \mathrm{e}^{-\boldsymbol{B}^{\mathrm{T}}}\,\hat{\mathcal{H}}\,\mathrm{e}^{-\boldsymbol{B}} = \begin{pmatrix} (g - B\,g^{-1}\,B)_{mn} & B_{mk}\,g^{kn} \\ -g^{mk}\,B_{kn} & g^{mn} \end{pmatrix}\,, \\
&\hat{\mathcal{H}} \equiv \begin{pmatrix} g_{mn} & 0 \\ 0 & g^{mn} \end{pmatrix}\,, \quad (\boldsymbol{B}^M{}_N) \equiv \begin{pmatrix} 0 & 0 \\ B_{mn} & 0 \end{pmatrix}\,, \quad \mathrm{e}^{\boldsymbol{B}} = \begin{pmatrix} \delta^m_n & 0 \\ B_{mn} & \delta^n_m \end{pmatrix}\,,
\end{aligned}
\tag{2.1}
$$

where g_{mn} and B_{mn} are the internal components of the metric and the Kalb-Ramond 2-form, respectively. As $\mathcal{H} \in \mathrm{O}(n,n)$, the non-linear transformations of the T-duality group are covariantly realized as

$$
\mathcal{H} \quad \to \quad \mathcal{O}^T \mathcal{H} \mathcal{O}\,, \qquad\qquad \mathcal{O} \in \mathrm{O}(n,n)\,. \tag{2.2}
$$

Let us now review some illustrative examples of T-folds that have been studied in the literature.

2.1 A toy example

We start by reviewing a toy model example that involves several duality transformations of a given background. This example has been discussed in [70, 71]. To be pedagogical and provide simple exercises, this subsection presents geometric cases like a twisted torus and a torus with H-flux before introducing a T-fold example.

[2] We can also have $T^n_{\text{phys}} = \tilde{T}^n$, which corresponds to a dual geometric description.
[3] These orbits have been determined in terms of a classification of gauged supergravities in [73].

JHEP12(2017)108

Twisted torus. Let us consider the metric of a twisted torus,

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + (\mathrm{d}z - m\,x\,\mathrm{d}y)^2\,, \qquad (m \in \mathbb{Z})\,. \tag{2.3}$$

Note that this is not a supergravity solution for $m \neq 0$, but still is a useful example to reveal a non-geometric global property. As this background has isometries along y and z directions, these directions can be compactified with certain boundary conditions. For example, let us take

$$(x,\,y,\,z) \sim (x,\,y+1,\,z)\,, \qquad (x,\,y,\,z) \sim (x,\,y,\,z+1)\,. \tag{2.4}$$

Apparently, there is no isometry along the x direction, but there actually exists a deformed Killing vector,

$$k = \partial_x + m\,y\,\partial_z\,. \tag{2.5}$$

Thus, this isometry direction can be compactified as

$$(x,\,y,\,z) \sim \mathrm{e}^{k}(x,\,y,\,z) = (x+1,\,y,\,z+m\,y)\,. \tag{2.6}$$

According to this identification, a 1-form $e_z \equiv \mathrm{d}z - m\,x\,\mathrm{d}y$ is globally well-defined [70], and the metric (2.3) is also globally well-defined.

When this background is regarded as a 2-torus $T^2_{y,z}$ fibered over a base S^1_x, the metric of the 2-torus takes the form

$$(g_{mn}) = \begin{pmatrix} 1 & -m\,x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -m\,x & 1 \end{pmatrix}\,. \tag{2.7}$$

Then, as one moves around the base S^1_x, the metric is transformed by a GL(2) rotation. That is to say, for $x \to x+1$, the metric is given by

$$g_{mn}(x+1) = \bigl[\Omega^{\mathrm{T}}\,g(x)\,\Omega\bigr]_{mn}\,, \qquad \Omega^m{}_n \equiv \begin{pmatrix} 1 & 0 \\ -m & 1 \end{pmatrix}\,. \tag{2.8}$$

This monodromy twist can be compensated by a coordinate transformation

$$y = y'\,, \qquad z = z' + m\,y'\,. \tag{2.9}$$

Thus the metric is single-valued up to the above coordinate transformation. Then this background can be understood to be geometric because general coordinate transformations belong to the gauge group of supergravity.

Torus with $\boldsymbol{H}$-flux. When a T-duality is formally performed on the twisted torus (2.3) along the x direction, we obtain the following background

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\,, \qquad B_2 = -m\,x\,\mathrm{d}y\wedge\mathrm{d}z\,, \tag{2.10}$$

equipped with the H-flux,

$$H_3 = \mathrm{d}B_2 = -m\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z\,. \tag{2.11}$$

If we consider the generalized metric (2.1) on the doubled torus $(y, z, \tilde{y}, \tilde{z})$ associated to this background, then we can easily identify the induced monodromy when $x \to x+1$. In this case, the monodromy matrix is given by

$$\mathcal{H}_{MN}(x+1) = \left[\Omega^{\mathrm{T}}\, \mathcal{H}(x)\, \Omega\right]_{MN}, \qquad \Omega^M{}_N = \begin{pmatrix} \delta^m_n & 0 \\ 2\, m\, \delta^y_{[m}\, \delta^z_{n]} & \delta^n_m \end{pmatrix} \in \mathrm{O}(2,2;\mathbb{Z})\,. \tag{2.12}$$

Then, the induced monodromy can be compensated by a constant shift in the B-field,

$$B_{yz} \quad \to \quad B_{yz} - m. \tag{2.13}$$

This shift transformation, which makes the background single-valued, belongs to the gauge transformations of supergravity. Hence we conclude that the background is geometric.

***T*-fold.** Finally, let us perform another T-duality transformation along the y-direction on the twisted torus (2.3). Then we obtain the following background [70]:

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \frac{\mathrm{d}y^2 + \mathrm{d}z^2}{1 + m^2\, x^2}\,, \qquad B_2 = \frac{m\, x}{1 + m^2\, x^2}\, \mathrm{d}y \wedge \mathrm{d}z\,. \tag{2.14}$$

In this case, neither general coordinate transformations nor B-field gauge transformations are enough to remove the multi-valuedness of the background. This can also be seen by calculating the monodromy matrix. The associated generalized metric is given by

$$\mathcal{H}(x) = \begin{pmatrix} \delta^p_m & 0 \\ -2\, m\, x\, \delta^{[m}_y\, \delta^{p]}_z & \delta^m_p \end{pmatrix} \begin{pmatrix} \delta_{pq} & 0 \\ 0 & \delta^{pq} \end{pmatrix} \begin{pmatrix} \delta^q_n & 2\, m\, x\, \delta^{[q}_y\, \delta^{n]}_z \\ 0 & \delta^n_q \end{pmatrix}. \tag{2.15}$$

Then, we find that, upon the transformation $x \to x+1$, the induced monodromy is

$$\mathcal{H}_{MN}(x+1) = \left[\Omega^{\mathrm{T}}\, \mathcal{H}(x)\, \Omega\right]_{MN}, \qquad \Omega^M{}_N \equiv \begin{pmatrix} \delta^m_n & 2\, m\, \delta^{[m}_y\, \delta^{n]}_z \\ 0 & \delta^n_m \end{pmatrix} \in \mathrm{O}(2,2;\mathbb{Z})\,. \tag{2.16}$$

The present $\mathrm{O}(2,2;\mathbb{Z})$ monodromy matrix Ω takes an upper-triangular form (called a β-transformation) which is not part of the gauge group of supergravity. Hence, to keep the background globally well defined, the transition functions that glue the local patches should be extended to the full set of $\mathrm{O}(2,2;\mathbb{Z})$ transformations beyond general coordinate transformations and B-field gauge transformations. This is what happens to the T-fold case.

In summary, we conclude that a non-geometric background with a non-trivial $\mathrm{O}(n,n;\mathbb{Z})$ monodromy transformation, such as a β-transformation, is a T-fold. The background (2.14) is a simple example.

From a viewpoint of DFT, by choosing a suitable solution of the section condition, the β-transformations can be realized as the gauge symmetries. Indeed, the above $\mathrm{O}(2,2;\mathbb{Z})$ monodromy matrix Ω can be canceled by a generalized coordinate transformation on the double torus coordinates $(y, z, \tilde{y}, \tilde{z})$,

$$y = y' + m\, \tilde{z}'\,, \qquad z = z'\,, \qquad \tilde{y} = \tilde{y}'\,, \qquad \tilde{z} = \tilde{z}'\,. \tag{2.17}$$

In this sense, the twisted doubled torus is globally well-defined in DFT.

In addition, it is also possible to make the single-valuedness manifest by introducing the dual fields G_{mn} and β^{mn} [66, 74–77] defined by

$$(G^{-1}+\beta)^{mn} \equiv (E^{-\mathrm{T}})^{mn}\,, \quad E_{mn} \equiv g_{mn} + B_{mn}\,, \tag{2.18}$$

or equivalently,

$$G_{mn} = E_{mk}\,E_{nl}\,g^{kl} = (g - B\,g^{-1}\,B)_{mn}\,, \qquad \beta^{mn} = (E^{-\mathrm{T}})^{mk}\,(E^{-\mathrm{T}})^{nl}\,B_{kl}\,. \tag{2.19}$$

The dual metric G_{mn} is precisely the same as the open-string metric [78], and the original metric g_{mn} may be called the closed-string metric. In terms of these fields, the generalized metric can be parameterized as (see for example [79])

$$\begin{gathered}\mathcal{H} = \mathrm{e}^{\boldsymbol{\beta}^{\mathrm{T}}}\,\check{\mathcal{H}}\,\mathrm{e}^{\boldsymbol{\beta}} = \begin{pmatrix} G_{mn} & G_{mk}\,\beta^{kn} \\ -\beta^{mk}\,G_{kn} & (G^{-1} - \beta\,G\,\beta)^{mn}\end{pmatrix}\,, \\ \check{\mathcal{H}} \equiv \begin{pmatrix} G_{mn} & 0 \\ 0 & G^{mn}\end{pmatrix}\,, \quad (\boldsymbol{\beta}^{M}{}_{N}) \equiv \begin{pmatrix} 0 & \beta^{mn} \\ 0 & 0\end{pmatrix}\,, \quad \mathrm{e}^{\boldsymbol{\beta}} = \begin{pmatrix} \delta^m_n & \beta^{mn} \\ 0 & \delta^n_m\end{pmatrix}\,,\end{gathered} \tag{2.20}$$

which is referred to as a non-geometric parameterization of the generalized metric. At the same time, the parameterization of the DFT dilaton is also changed by introducing the dual dilaton $\tilde{\phi}$,

$$\mathrm{e}^{-2d} = \mathrm{e}^{-2\tilde{\phi}}\,\sqrt{|G|}\,. \tag{2.21}$$

In the non-geometric parameterization, the background (2.15) becomes

$$\mathrm{d}s^2_{\text{dual}} \equiv G_{mn}\,\mathrm{d}x^m\,\mathrm{d}x^n = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\,, \qquad \beta^{yz} = m\,x\,, \tag{2.22}$$

and the $\mathrm{O}(2,2;\mathbb{Z})$ monodromy matrix (2.16) corresponds to a constant shift in the β field; $\beta^{yz} \to \beta^{yz} + m$. Namely, up to a constant β-shift, which is a gauge symmetry (2.17) of DFT, the background becomes single-valued.

In this paper, we define a non-geometric Q-flux as [80]

$$Q_p{}^{mn} \equiv \partial_p \beta^{mn}\,. \tag{2.23}$$

Then, upon a transformation $x \to x+1$, the induced monodromy on the β-field is measured by an integral of the Q-flux,

$$\beta^{mn}(x+1) - \beta^{mn}(x) = \int_x^{x+1} \mathrm{d}x'^p\,\partial_p\beta^{mn}(x') = \int_x^{x+1} \mathrm{d}x'^p\,Q_p{}^{mn}(x')\,. \tag{2.24}$$

This expression plays the central role in our argument.

After this illustrative example we conclude that Q-flux backgrounds are globally well-defined as T-folds. In the next subsection, let us explain a codimension-1 example of the exotic 5^2_2-brane by using the above Q-flux.

2.2 Codimension-1 5^2_2-brane background

The second example is a supergravity solution studied in [72]. It is obtained by smearing the codimension-2 exotic 5^2_2-brane solution [81, 82], which is related to the NS5-brane solution by two T-duality transformations. It is also referred to as a Q-brane, as it is a source of Q-flux, as we are going to check. The codimension-1 version of this solution is given by

$$\begin{aligned} \mathrm{d}s^2 &= m\,x\,(\mathrm{d}x^2+\mathrm{d}y^2)+\frac{x\,(\mathrm{d}z^2+\mathrm{d}w^2)}{m\,(x^2+z^2)}+\mathrm{d}s^2_{\mathbb{R}^6}\,, \\ B_2 &= \frac{x}{m\,(x^2+z^2)}\mathrm{d}z\wedge\mathrm{d}w\,, \qquad \Phi=\frac{1}{2}\ln\biggl[\frac{x}{m\,(x^2+z^2)}\biggr]\,. \end{aligned} \tag{2.25}$$

With the non-geometric parameterization (2.20), this solution is simplified as

$$\begin{aligned} \mathrm{d}s^2_{\text{dual}} &= m\,x\,(\mathrm{d}x^2+\mathrm{d}y^2)+\frac{\mathrm{d}z^2+\mathrm{d}w^2}{m\,x}+\mathrm{d}s^2_{\mathbb{R}^6}\,, \\ \beta^{zw} &= m\,y\,, \qquad \tilde{\phi}=\frac{1}{2}\ln\biggl[\frac{1}{m\,x}\biggr]\,. \end{aligned} \tag{2.26}$$

Assuming that the y direction is compactified with $y\sim y+1$, the monodromy under $y\to y+1$ is given by a constant β-shift;

$$\beta^{zw}\to\beta^{zw}+m\,. \tag{2.27}$$

As the background is twisted by a β-shift, this example can be considered as a T-fold. In terms of the Q-flux, this solution has a constant Q-flux,

$$Q_y{}^{zw}=m\,. \tag{2.28}$$

Finally, the monodromy matrix is given by

$$\mathcal{H}_{MN}(y+1)=\bigl[\Omega^{\mathrm{T}}\,\mathcal{H}(y)\,\Omega\bigr]_{MN}\,, \qquad \Omega^M{}_N\equiv\begin{pmatrix}\delta^m_n & 2\,m\,\delta^{[m}_z\,\delta^{n]}_w\\ 0 & \delta^n_m\end{pmatrix}\in\mathrm{O}(10,10;\mathbb{Z})\,. \tag{2.29}$$

By employing the knowledge on T-folds introduced in this section, we will elaborate on a non-geometric aspect of YB-deformed backgrounds as T-folds.

3 Non-geometric aspects of YB deformations

Let us show that various YB-deformed backgrounds can be regarded as T-folds.

Subsection 3.1 is devoted to a brief review of the generalized supergravity to fix our convention and notation. In subsection 3.2, we explain how the homogeneous Yang-Baxter deformations are interpreted as β-twists and how a YB-deformed background can be derived from a given classical r-matrix. In subsection 3.3, the general structure of T-duality monodromy is revealed for the YB-deformed backgrounds studied in this paper. In subsection 3.4, various T-folds are obtained as YB-deformations of Minkowski spacetime. In subsection 3.5, we study a certain background which is obtained by a non-Abelian T-duality but is not described as a Yang-Baxter deformation. It is shown that this background is a solution of GSE and can also be regarded as a T-fold. In section 3.6, in order to study a more non-trivial class of T-folds with R-R fields, we consider some backgrounds obtained as YB-deformations of $\mathrm{AdS}_5\times\mathrm{S}^5$.

JHEP12(2017)108

3.1 Generalized supergravity

The generalized type IIB supergravity equations of motion were originally derived in [64, 65]. Just for later convenience, we will follow the convention utilized in [32] hereafter.

Then the generalized type II supergravity equations of motion are given by

$$
\begin{aligned}
R_{mn}-\frac{1}{4}H_{mpq}H_n{}^{pq}+2D_m\partial_n\Phi+D_mU_n+D_nU_m=T_{mn}\,,\\
R+4D^m\partial_m\Phi-4|\partial\Phi|^2-\frac{1}{2}|H_3|^2-4\big(I^mI_m+U^mU_m+2U^m\partial_m\Phi-D_mU^m\big)=0,\\
-\frac{1}{2}D^kH_{kmn}+\partial_k\Phi H^k{}_{mn}+U^kH_{kmn}+D_mI_n-D_nI_m=\mathcal{K}_{mn}\,,\\
\mathrm{d}*\hat{\mathsf{F}}_p-H_3\wedge*\hat{\mathsf{F}}_{p+2}-\iota_IB_2\wedge*\hat{\mathsf{F}}_p-\iota_I*\hat{\mathsf{F}}_{p-2}=0,
\end{aligned}
\tag{3.1}
$$

where $I = I^m\,\partial_m$ is a Killing vector satisfying

$$
\pounds_I g_{mn} = D_mI_n + D_nI_m = 0\,. \tag{3.2}
$$

Here D_m is the covariant derivative associated with the metric g_{mn}, $*$ is the Hodge star operator, and ι_I is the interior product with the vector I. In addition, we have introduced the following quantities:

$$
\begin{aligned}
T_{mn} &\equiv \frac{1}{4}\,\mathrm{e}^{2\Phi}\sum_p\left[\frac{1}{(p-1)!}\,\hat{\mathsf{F}}_{(m}{}^{k_1\cdots k_{p-1}}\hat{\mathsf{F}}_{n)k_1\cdots k_{p-1}}-\frac{1}{2}\,g_{mn}\,|\hat{\mathsf{F}}_p|^2\right],\\
\mathcal{K}_{mn} &\equiv \frac{1}{4}\,\mathrm{e}^{2\Phi}\sum_p\frac{1}{(p-2)!}\,\hat{\mathsf{F}}_{k_1\cdots k_{p-2}}\,\hat{\mathsf{F}}_{mn}{}^{k_1\cdots k_{p-2}}\,,\qquad U_m\equiv I^n\,B_{nm}\,.
\end{aligned}
\tag{3.3}
$$

Here, $0 \le p \le 9$ takes an even/odd number for type IIA/IIB theory, respectively. The R-R field strengths should satisfy the self-duality relation,

$$
\,\hat{\mathsf{F}}_p = (-1)^{\frac{p(p+1)}{2}+1}\hat{\mathsf{F}}_{10-p}\,,\qquad \hat{\mathsf{F}}_p = (-1)^{\frac{p(p-1)}{2}}\hat{\mathsf{F}}_{10-p}\,. \tag{3.4}
$$

Given the R-R field strengths, the R-R potentials can be determined through the relation,

$$
\hat{\mathsf{F}}_p = \mathrm{d}\hat{\mathsf{C}}_{p-1} + H_3\wedge\hat{\mathsf{C}}_{p-3} - \iota_IB_2\wedge\hat{\mathsf{C}}_{p-1} - \iota_I\hat{\mathsf{C}}_{p+1}\,. \tag{3.5}
$$

Note that when $I = 0$, the above expressions reduce to those of the usual supergravity.

It is also convenient to define the R-R fields $(\mathsf{F},\,\mathsf{A})$ and $(\check{\mathsf{F}},\,\check{\mathsf{C}})$ as

$$
\mathsf{F}\equiv\mathrm{e}^{B_2\wedge}\,\hat{\mathsf{F}}\,,\qquad \mathsf{A}\equiv\mathrm{e}^{B_2\wedge}\,\hat{\mathsf{C}}\,,\qquad \check{\mathsf{F}}\equiv\mathrm{e}^{\beta\vee}\,\mathsf{F}\,,\qquad \check{\mathsf{C}}\equiv\mathrm{e}^{\beta\vee}\,\mathsf{A}\,, \tag{3.6}
$$

satisfying

$$
\begin{aligned}
\mathsf{F}_p &= \mathrm{d}\mathsf{A}_{p-1} - \iota_I\mathsf{A}_{p+1}\,,\\
\check{\mathsf{F}}_p &= \mathrm{d}\check{\mathsf{C}}_{p-1} - \frac{1}{2}\,Q^{mn}\wedge\iota_m\iota_n\check{\mathsf{C}}_{p+1} - \iota_I\check{\mathsf{C}}_{p+1}\qquad \big(Q^{mn}\equiv Q_k{}^{mn}\,\mathrm{d}x^k\big)\,.
\end{aligned}
\tag{3.7}
$$

Here, for a bi-vector β^{mn} and a p-form α_p, we have defined

$$
\beta\vee\alpha_p \equiv \frac{1}{2}\,\beta^{mn}\,\iota_m\,\iota_n\alpha_p\,. \tag{3.8}
$$

In order to distinguish three definitions of R-R fields, we call $(\hat{\mathsf{F}},\,\hat{\mathsf{C}})$ B-untwisted R-R fields while $(\check{\mathsf{F}},\,\check{\mathsf{C}})$ β-untwisted R-R fields,

$$(\check{\mathsf{F}},\,\check{\mathsf{C}}) \quad \xleftarrow[\beta\text{-untwist}]{\ \mathrm{e}^{\beta\vee}\ } \quad (\mathsf{F},\,\mathsf{A}) \quad \xrightarrow[B\text{-untwist}]{\ \mathrm{e}^{-B_2\wedge}\ } \quad (\hat{\mathsf{F}},\,\hat{\mathsf{C}})\,. \tag{3.9}$$

Following the same terminology, we call the dual metric and the dual dilaton $(G_{mn},\,\tilde{\phi})$ the β-untwisted fields,

$$\begin{array}{ccccc} & & \mathcal{H} = \mathrm{e}^{-\boldsymbol{B}^{\mathrm{T}}}\,\hat{\mathcal{H}}\,\mathrm{e}^{-\boldsymbol{B}} & \xrightarrow[B\text{-untwist}]{} & \hat{\mathcal{H}} = \begin{pmatrix} g_{mn} & 0 \\ 0 & g^{mn} \end{pmatrix} \\ \check{\mathcal{H}} = \begin{pmatrix} G_{mn} & 0 \\ 0 & G^{mn} \end{pmatrix} & \xleftarrow[\beta\text{-untwist}]{} & = \mathrm{e}^{\boldsymbol{\beta}^{\mathrm{T}}}\,\check{\mathcal{H}}\,\mathrm{e}^{\boldsymbol{\beta}} & & \\ \mathrm{e}^{-2\tilde{\phi}} & \xleftarrow[\beta\text{-untwist}]{} & \sqrt{|G|}\,\mathrm{e}^{-2\tilde{\phi}} = \mathrm{e}^{-2d} = \sqrt{|g|}\,\mathrm{e}^{-2\Phi} & \xrightarrow[B\text{-untwist}]{} & \mathrm{e}^{-2\Phi}\,. \end{array} \tag{3.10}$$

The B-untwisted fields are invariant under B-field gauge transformations while the β-untwisted fields are invariant under β-transformations.

When $I = 0$, the B-untwisted fields $(\hat{\mathsf{F}},\,\hat{\mathsf{C}},\,g_{mn},\,\Phi)$ together with B_{mn} are frequently utilized in some contexts. For example, these are the background fields appearing in the Green-Schwarz superstring action. On the other hand, the β-untwisted fields $(\check{\mathsf{F}},\,\check{\mathsf{C}},\,G_{mn},\,\tilde{\phi})$ are unfamiliar quantities but play an important role in the context of YB deformation. To study the monodromy of T-folds, the objects in the middle, i.e., $(\mathsf{F},\,\mathsf{A},\,\mathcal{H}_{MN},\,d)$ will play an important role, as we will discuss later.

Before closing this subsection, it is worth noting the divergence formula observed in [25, 26, 33]. For the solutions of GSE obtained as YB deformations and a non-Abelian T-duality discussed in this paper, the Killing vector I can always be found from the following formula:

$$I^m = \tilde{D}_n \beta^{mn}\,, \tag{3.11}$$

where $\tilde{D}$ is associated with the β-untwisted metric g_{mn} . The general proof of this expression for the general YB deformations based on the mCYBE and the homogeneous CYBE will be reported in the coming paper [83].

3.2 YB deformations as β-deformations

YB deformations of type IIB string theory on $\mathrm{AdS}_5\times\mathrm{S}^5$ have been presented in [10, 12]. It used to be quite a difficult problem to read off the full expression of YB deformed background, because it is necessary to perform supercoset construction but it is really complicated and the computation becomes messy.

In the pioneering work [42], the supercoset construction was done for the q-deformed $\mathrm{AdS}_5\times\mathrm{S}^5$. Then the technique was generalized to the homogeneous CYBE case in [52]. After these developments, this technique was refined in [55] based on κ-symmetry. In the recent paper [33],[4] a much simpler way has been proposed. This is a direct formula between the fields in GSE and classical r-matrices satisfying the homogeneous CYBE and relies on

[4]A similar way has been elaborated also from the viewpoint of the invariance of the Page form [84].

the divergence formula (3.11). In the following, we will give a brief review of this simple formula and explain how to use it by taking a simple example.

As we introduced in section 2, the β-deformations (or the β-transformations) belong to a particular class of $\mathrm{O}(D,D)$ transformations under which the β-field is shifted as

$$\beta^{mn}(x) \to \beta'^{mn}(x) = \beta^{mn}(x) + r^{mn}(x) \qquad (r^{mn} = -r^{nm}), \tag{3.12}$$

while the β-untwisted fields remain invariant,

$$\check{\mathcal{H}}' = \check{\mathcal{H}}, \qquad \tilde{\phi}' = \tilde{\phi}, \qquad \check{\mathsf{F}}' = \check{\mathsf{F}}, \qquad \check{\mathsf{C}}' = \check{\mathsf{C}}. \tag{3.13}$$

Unlike the B-field gauge transformations, the β-deformation is not a gauge transformation, and in general, the β-deformed background may not satisfy the (generalized) supergravity equations (3.1) even if the original background is a solution of the supergravity (or DFT).

Now, let us explain a relation between the β-deformation and the YB deformation. For this purpose, we concentrate on deformations of a background with vanishing B-field, and then the β-field in the original background also vanishes. A homogeneous YB deformation is specified by taking a skew-symmetric classical r-matrix

$$r = \frac{1}{2}\, r^{ij}\, T_i \wedge T_j = r^{ij}\, T_i \otimes T_j\,, \qquad r^{ij} = -r^{ji}\,, \tag{3.14}$$

which satisfies the homogeneous classical Yang-Baxter equation (CYBE),

$$f_{l_1 l_2}{}^{i}\, r^{j l_1}\, r^{k l_2} + f_{l_1 l_2}{}^{j}\, r^{k l_1}\, r^{i l_2} + f_{l_1 l_2}{}^{k}\, r^{i l_1}\, r^{j l_2} = 0\,. \tag{3.15}$$

Here r^{ij} is a constant skew-symmetric matrix and T_i's are the elements of the Lie algebra $\mathfrak{g}$ associated with the bosonic isometry group G, satisfying commutation relations

$$[T_i, T_j] = f_{ij}{}^{k}\, T_k \qquad (f_{ij}{}^{k}\ :\ \text{the structure constant})\,. \tag{3.16}$$

An important observation made in [33] is that a YB-deformed background associated with the classical r-matrix (3.14) can also be generated by a β-deformation,

$$\beta^{mn}(x) \to \beta^{mn}(x) + r^{mn}(x)\,, \qquad \frac{1}{2}\, r^{mn}(x)\, \partial_m \wedge \partial_n \equiv \eta\, r^{ij}\, \hat{T}_i(x) \wedge \hat{T}_j(x)\,. \tag{3.17}$$

Here, a real constant η is a deformation parameter and $\hat{T}_i$ are Killing vector fields on the original background satisfying the same commutation relations (3.16). Since $\beta^{mn} = 0$ in the undeformed background, we obtain the following expression

$$\beta^{(r)mn}(x) = r^{mn}(x) = 2\,\eta\, r^{ij}\, \hat{T}_i^m(x)\, \hat{T}_j^n(x) \tag{3.18}$$

for the YB-deformed background.

In terms of the usual supergravity fields (g_{mn}, B_{mn}, Φ, $\hat{\mathsf{F}}$, $\hat{\mathsf{C}}$), the YB-deformed background can be expressed as

$$\begin{aligned} g^{(r)}_{mn} + B^{(r)}_{mn} &= \big[(G^{-1} - \beta^{(r)})\big]^{-1}_{mn}\,, \\ \mathrm{e}^{-2\Phi^{(r)}} &= \mathrm{e}^{-2\tilde{\phi}}\sqrt{\det[\delta^m_n - (G\,\beta^{(r)}\,G\,\beta^{(r)})_m{}^n]}\,, \\ \hat{\mathsf{F}}^{(r)} &= \mathrm{e}^{-B^{(r)}_2\wedge}\,\mathrm{e}^{-\beta^{(r)}\vee}\,\check{\mathsf{F}}\,, \qquad \hat{\mathsf{C}}^{(r)} = \mathrm{e}^{-B^{(r)}_2\wedge}\,\mathrm{e}^{-\beta^{(r)}\vee}\,\check{\mathsf{C}}\,, \end{aligned} \tag{3.19}$$

where the β-untwisted fields ($G_{mn} = g_{mn}$, $\tilde{\phi} = \Phi$, $\check{\mathsf{F}} = \hat{\mathsf{F}}$, $\check{\mathsf{C}} = \hat{\mathsf{C}}$) are the original undeformed background with $B_2 = 0$. The deformed background solves the (generalized) supergravity equations of motion (3.1). In this way, we can generate YB-deformed backgrounds by using the formula (3.19) with the β-field (3.18).

Furthermore, it is interesting to note that the homogeneous CYBE (3.15) can also be expressed as

$$R \equiv [\beta^{(r)},\, \beta^{(r)}]_S = 0\,, \tag{3.20}$$

where $[\,,\,]_S$ denotes the Schouten bracket and the tri-vector R is known as the non-geometric R-flux. The Schouten bracket is defined for a p-vector and a q-vector as

$$\begin{aligned}&[a_1 \wedge \cdots \wedge a_p,\, b_1 \wedge \cdots \wedge b_q]_{\mathrm{S}}\\ &\quad\equiv \sum_{i,j} (-1)^{i+j} [a_i,\, b_j] \wedge a_1 \wedge \cdots \check{a}_i \cdots \wedge a_p \wedge b_1 \wedge \cdots \check{b}_j \cdots \wedge b_q\,,\end{aligned} \tag{3.21}$$

where the check $\check{a}_i$ denotes the omission of a_i. This fact implies that the non-geometric R-flux vanishes for the homogeneous YB-deformed backgrounds (as far as the undeformed background has vanishing B-field).

Minkowski and $\mathbf{AdS_5 \times S^5}$ backgrounds. In the following subsections, we will consider YB deformations of 10D Minkowski spacetime and the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. Before presenting various examples, we will introduce the coordinate systems and show the explicit form of the Killing vector fields $\hat{T}_i$ in 10D Minkowski spacetime and the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background.

For a 10D Minkowski spacetime, we take the standard Minkowski metric,

$$\mathrm{d}s^2_{\mathrm{Min}} = \eta_{mn}\, \mathrm{d}x^m\, \mathrm{d}x^n \qquad (m, n = 0, 1, \ldots, 9)\,, \tag{3.22}$$

where $\eta_{mn} = \mathrm{diag}(-1, +1, \ldots, +1)$. In this coordinate system, the Killing vector fields $\{\hat{T}_i\} = \{\hat{P}_m,\, \hat{M}_{mn}\}$ are expressed as

$$\hat{P}_m = -\partial_m\,, \qquad \hat{M}_{mn} = x_m\, \partial_n - x_n\, \partial_m\,. \tag{3.23}$$

These vector fields realize the following Poincaré algebra:

$$\begin{aligned}[P_m,\, M_{nk}] &= \eta_{mn}\, P_k - \eta_{mk}\, P_n\,,\\ [M_{mn},\, M_{kl}] &= -\eta_{mk}\, M_{nl} + \eta_{nk}\, M_{ml} + \eta_{ml}\, M_{nk} - \eta_{nl}\, M_{mk}\,.\end{aligned} \tag{3.24}$$

Here P_m and M_{mn} are the translation and Lorentz generators of the Poincaré group $\mathrm{ISO}(1,9)$.

When we consider the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background as the original background, we choose the following coordinate system:

$$\begin{aligned}\mathrm{d}s^2_{\mathrm{AdS}_5 \times \mathrm{S}^5} &= \mathrm{d}s^2_{\mathrm{AdS}_5} + \mathrm{d}s^2_{\mathrm{S}^5}\,,\\ \mathrm{d}s^2_{\mathrm{AdS}_5} &= \frac{\mathrm{d}z^2 - (\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2}{z^2}\,,\\ \mathrm{d}s^2_{\mathrm{S}^5} &= \mathrm{d}r^2 + \sin^2 r\, \mathrm{d}\xi^2 + \cos^2\xi\, \sin^2 r\, \mathrm{d}\phi_1^2 + \sin^2 r\, \sin^2\xi\, \mathrm{d}\phi_2^2 + \cos^2 r\, \mathrm{d}\phi_3^2\,.\end{aligned} \tag{3.25}$$

JHEP12(2017)108

The R-R 5-form field strength in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background is given by

$$\hat{\mathsf{F}}_5 = 4\left(\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}\right), \qquad \omega_{\mathrm{AdS}_5} = \bar{*}\,\omega_{\mathrm{S}^5}\,, \tag{3.26}$$

where the volume forms ω_{AdS_5} and ω_{S^5} are defined as, respectively,

$$\begin{aligned} \omega_{\mathrm{AdS}_5} &= -\frac{\mathrm{d}z\wedge\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3}{z^5}\,, \\ \omega_{\mathrm{S}^5} &= \sin^3 r\cos r\sin\xi\cos\xi\,\mathrm{d}r\wedge\mathrm{d}\xi\wedge\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2\wedge\mathrm{d}\phi_3\,, \end{aligned} \tag{3.27}$$

and $\bar{*}$ is the Hodge star operator associated with the undeformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ background,

$$(\bar{*}\alpha_p)_{n_1\cdots n_2n_3n_4n_{10-p}} = \frac{1}{p!}\,\varepsilon^{m_1\cdots m_p}{}_{n_1\cdots n_2n_3n_4n_{10-p}}\,\alpha_{m_1\cdots m_p}\,, \qquad \varepsilon_{z0123r\xi\phi_1\phi_2\phi_3} = +\sqrt{|g|}\,. \tag{3.28}$$

It is also convenient to define ω_4 as

$$\omega_4 \equiv \sin^4 r\sin\xi\cos\xi\,\mathrm{d}\xi\wedge\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2\wedge\mathrm{d}\phi_3\,. \tag{3.29}$$

Note that $\mathrm{d}\omega_4 = 4\,\omega_{\mathrm{S}^5}$.

The non-vanishing commutation relations for the isometry group SO(2, 4) of AdS_5 are

$$\begin{aligned} [P_\mu,\,K_\nu] &= 2\,(M_{\mu\nu} + \eta_{\mu\nu}\,D)\,, \qquad [D,\,P_\mu] = P_\mu\,, \quad [D,\,K_\mu] = -K_\mu\,, \\ [P_\mu,\,M_{\nu\rho}] &= \eta_{\mu\nu}\,P_\rho - \eta_{\mu\rho}\,P_\nu\,, \quad [K_\mu,\,M_{\nu\rho}] = \eta_{\mu\nu}\,K_\rho - \eta_{\mu\rho}\,K_\nu\,, \\ [M_{\mu\nu},\,M_{\rho\sigma}] &= \eta_{\mu\sigma}\,M_{\nu\rho} + \eta_{\nu\rho}\,M_{\mu\sigma} - \eta_{\mu\rho}\,M_{\nu\sigma} - \eta_{\nu\sigma}\,M_{\mu\rho}\,, \end{aligned} \tag{3.30}$$

where $\mu\,,\nu = 0,1,2,3$, and D, and K_μ are generators of the dilatation and special conformal transformations, respectively. The commutation relations (3.30) are realized by the following Killing vector fields, $\{\hat{T}_i\} = \{\hat{P}_m,\,\hat{K}_m,\,\hat{M}_{mn},\,\hat{D}\}$:

$$\begin{aligned} \hat{P}_\mu &= -\partial_\mu\,, \qquad \hat{K}_\mu = -(z^2 + x_\nu\,x^\nu)\,\partial_\mu + 2\,x_\mu\,(z\,\partial_z + x^\nu\,\partial_\nu)\,, \\ \hat{M}_{\mu\nu} &= x_\mu\,\partial_\nu - x_\nu\,\partial_\mu\,, \quad \hat{D} = -z\,\partial_z - x^\mu\,\partial_\mu\,. \end{aligned} \tag{3.31}$$

We will use the above Killing vector fields (3.24), (3.31) to obtain an explicit expression of the β-field $\beta^{(r)} = \eta\,r^{ij}\,\hat{T}_i\wedge\hat{T}_j$ associated with a given r-matrix $r = \frac{1}{2}\,r^{ij}\,T_i\wedge T_j$. In the following, we will omit the superscript ${}^{(r)}$ for the YB-deformed backgrounds.

An example: the Maldacena-Russo background. To demonstrate how to use the formula (3.19), let us consider a YB-deformed $\mathrm{AdS}_5 \times S^5$ background associated with a classical r-matrix [23],

$$r = \frac{1}{2}\,P_1\wedge P_2\,. \tag{3.32}$$

This r-matrix is Abelian and satisfies the homogeneous CYBE (3.15). The associated YB deformed background is derived in [23, 52].

The classical r-matrix (3.32) leads to the associated β-field,

$$\beta = \eta\,\hat{P}_1\wedge\hat{P}_2 = \eta\,\partial_1\wedge\partial_2\,. \tag{3.33}$$

JHEP12(2017)108

Then, the AdS_5 part of a 10×10 matrix $(G^{-1}-\beta)$ is

$$(G^{-1}-\beta)^{mn} = \begin{pmatrix} z^2 & 0 & 0 & 0 & 0 \\ 0 & -z^2 & 0 & 0 & 0 \\ 0 & 0 & z^2 & -\eta & 0 \\ 0 & 0 & \eta & z^2 & 0 \\ 0 & 0 & 0 & 0 & z^2 \end{pmatrix} , \tag{3.34}$$

where we have ordered the coordinates as $(z\,,x^0\,,x^1\,,x^2\,,x^3)$. By using the inverse of the matrix (3.34) and the formula (3.19), we obtain the NS-NS fields of the YB-deformed background,

$$\begin{aligned} ds^2 &= \frac{dz^2-(dx^0)^2+(dx^3)^2}{z^2} + \frac{z^2\,[(dx^1)^2+(dx^2)^2]}{z^4+\eta^2} + ds^2_{\mathrm{S}^5}\,, \\ B_2 &= \frac{\eta}{z^4+\eta^2}\,dx^1\wedge dx^2\,, \qquad \Phi = \frac{1}{2}\,\ln\left[\frac{z^4}{z^4+\eta^2}\right]. \end{aligned} \tag{3.35}$$

The next task is to derive the R-R fields of the deformed background. From the undeformed R-R 5-form field strength (3.26) of the $\mathrm{AdS}_5\times \mathrm{S}^5$ background, the R-R fields F are given by

$$\begin{aligned} \mathsf{F} = e^{-\beta\vee}\,\check{\mathsf{F}}^{(0)} &= 4\left(\omega_{\mathrm{AdS}_5}+\omega_{\mathrm{S}^5}\right) - 4\,\beta\vee\omega_{\mathrm{AdS}_5} \\ &= 4\left(\omega_{\mathrm{AdS}_5}+\omega_{\mathrm{S}^5}\right) - 4\,\eta\,\frac{dz\wedge dx^0\wedge dx^3}{z^5}\,. \end{aligned} \tag{3.36}$$

This is nothing but a linear combination of the deformed R-R field strengths with different rank. Hence we can readily read off the following expressions:

$$\mathsf{F}_3 = -4\,\eta\,\frac{dz\wedge dx^0\wedge dx^3}{z^5}\,, \qquad \mathsf{F}_5 = 4\left(\omega_{\mathrm{AdS}_5}+\omega_{\mathrm{S}^5}\right). \tag{3.37}$$

Furthermore, the B-untwisted R-R fields $\hat{\mathsf{F}}$ can be computed as

$$\begin{aligned} \hat{\mathsf{F}} &= e^{-B_2\wedge}\,\mathsf{F} \\ &= -4\,\eta\,\frac{dz\wedge dx^0\wedge dx^3}{z^5} + 4\left(\frac{z^4}{z^4+\eta^2}\,\omega_{\mathrm{AdS}_5}+\omega_{\mathrm{S}^5}\right) - 4\,B_2\wedge\omega_{\mathrm{S}^5}\,. \end{aligned} \tag{3.38}$$

Namely, we obtain

$$\begin{aligned} &\hat{\mathsf{F}}_1 = 0\,, \qquad \hat{\mathsf{F}}_3 = -4\,\eta\,\frac{dz\wedge dx^0\wedge dx^3}{z^5}\,, \\ &\hat{\mathsf{F}}_5 = 4\left(\frac{z^4}{z^4+\eta^2}\,\omega_{\mathrm{AdS}_5}+\omega_{\mathrm{S}^5}\right), \\ &\hat{\mathsf{F}}_7 = -4\,B_2\wedge\omega_{\mathrm{S}^5}\,. \end{aligned} \tag{3.39}$$

The full deformed background, given by (3.35) and (3.39), is a solution of the standard type IIB supergravity. This background is nothing but a gravity dual of non-commutative gauge theory [59, 60].

Thus, nowadays, we do not have to perform supercoset construction to obtain the full expression of YB-deformed background. Just by using a simple formula (3.19), given a classical r-matrix, the full background can easily be derived.

JHEP12(2017)108

3.3 T-duality monodromy of YB-deformed background

As we explained in the previous subsection, the YB-deformed background described by $(\mathcal{H},\, d,\, \mathsf{F})$ always has the following structure:

$$
\begin{aligned}
&\mathcal{H} = \mathrm{e}^{\boldsymbol{r}^{\mathrm{T}}}\, \check{\mathcal{H}}^{(0)}\, \mathrm{e}^{\boldsymbol{r}}\,, \qquad\qquad d = d^{(0)}\,, \qquad\qquad \mathsf{F} = \mathrm{e}^{-r\vee}\, \check{\mathsf{F}}^{(0)}\,, \\
&\boldsymbol{r} \equiv \mathrm{e}^{\boldsymbol{\beta}} = \begin{pmatrix} \delta^m_n & r^{mn} \\ 0 & \delta_m^n \end{pmatrix}\,, \quad r^{mn} \equiv 2\,\eta\, r^{ij}\, \hat{T}^m_i\, \hat{T}^n_j\,,
\end{aligned}
\tag{3.40}
$$

where $(\check{\mathcal{H}}^{(0)},\, d^{(0)},\, \check{\mathsf{F}}^{(0)})$ represent the undeformed background. In the following examples, B-field vanishes in the undeformed background, and the β-field in the YB-deformed background is given by

$$
\beta = \frac{1}{2}\, r^{mn}\, \partial_m \wedge \partial_n = \eta\, r^{ij}\, \hat{T}_i \wedge \hat{T}_j\,. \tag{3.41}
$$

At this stage, we know only the local property of the YB-deformed background.

In the examples considered in this paper, the bi-vector r^{mn} (or the β-field in the YB-deformed background) always has a linear-coordinate dependence. Suppose that r^{mn} depends on a coordinate y linearly like,

$$
r^{mn} = \mathsf{r}^{mn}\, y + \bar{\mathsf{r}}^{mn} \qquad (\mathsf{r}^{mn}:\ \text{constant},\quad \bar{\mathsf{r}}^{mn}:\ \text{independent of } y)\,, \tag{3.42}
$$

and the β-untwisted fields are independent of y. Then, from the Abelian property,

$$
\mathrm{e}^{\boldsymbol{r}_1 + \boldsymbol{r}_2} = \mathrm{e}^{\boldsymbol{r}_1}\, \mathrm{e}^{\boldsymbol{r}_2} = \mathrm{e}^{\boldsymbol{r}_2}\, \mathrm{e}^{\boldsymbol{r}_1}\,, \qquad \mathrm{e}^{-(r_1+r_2)\vee} = \mathrm{e}^{-r_1\vee}\, \mathrm{e}^{-r_2\vee} = \mathrm{e}^{-r_2\vee}\, \mathrm{e}^{-r_1\vee}\,, \tag{3.43}
$$

we obtain

$$
\begin{aligned}
&\mathcal{H}_{MN}(y+a) = \bigl[\Omega_a^{\mathrm{T}} \mathcal{H}(y)\, \Omega_a\bigr]_{MN}\,, \quad d(y+a) = d(y)\,, \quad \mathsf{F}(y+a) = \mathrm{e}^{-\omega_a\vee}\, \check{\mathsf{F}}(y)\,, \\
&(\Omega_a)^M{}_N \equiv \begin{pmatrix} \delta^m_n & a\, \mathsf{r}^{mn} \\ 0 & \delta_m^n \end{pmatrix}\,, \qquad (\omega_a)^{mn} \equiv a\, \mathsf{r}^{mn}\,.
\end{aligned}
\tag{3.44}
$$

If we can find out an a_0 (where the $\mathrm{O}(10,10;\mathbb{R})$ matrix Ω_{a_0} is an element of $\mathrm{O}(10,10;\mathbb{Z})$), the background allows us to compactify the y direction as $y \sim y + a_0$. This is because $\mathrm{O}(10,10;\mathbb{Z})$ is a gauge symmetry of String Theory and the background can be identified up to the gauge transformation. In this example of T-fold, the monodromy matrices for the generalized metric and R-R fields are Ω_{a_0} and $\mathrm{e}^{-\omega_{a_0}\vee}$, respectively, while the dilaton d is single-valued. Note that the R-R potential A has the same monodromy as F.

3.4 YB-deformed Minkowski backgrounds

In this subsection, we study YB-deformations of Minkowski spacetime [85, 86]. We begin by a simple example of the Abelian YB deformation. Then two purely NS-NS solutions of GSE are presented and are shown to be T-folds. These backgrounds have vanishing R-R fields and are the first examples of purely NS-NS solutions of GSE.

JHEP12(2017)108

3.4.1 Abelian example

Let us consider a simple Abelian r-matrix [85]

$$r = -\frac{1}{2}\,P_1\wedge M_{23}\,. \tag{3.45}$$

The corresponding YB-deformed background becomes

$$\begin{aligned}
\mathrm{d}s^2 &= -(\mathrm{d}x^0)^2 + \frac{(\mathrm{d}x^1)^2 + \bigl[1+(\eta\,x^2)^2\bigr]\,(\mathrm{d}x^2)^2 + \bigl[1+(\eta\,x^3)^2\bigr]\,(\mathrm{d}x^3)^2 + 2\,\eta^2\,x^2\,x^3\,\mathrm{d}x^2\,\mathrm{d}x^3}{1+\eta^2\,\bigl[(x^2)^2+(x^3)^2\bigr]} \\
&\quad + \sum_{i=4}^{9}(\mathrm{d}x^i)^2\,, \\
B_2 &= \frac{\eta\,\mathrm{d}x^1\wedge\bigl(x^2\,\mathrm{d}x^3 - x^3\,\mathrm{d}x^2\bigr)}{1+\eta^2\,\bigl[(x^2)^2+(x^3)^2\bigr]}\,,\qquad \Phi = \frac{1}{2}\ln\biggl[\frac{1}{1+\eta^2\,\bigl[(x^2)^2+(x^3)^2\bigr]}\biggr]\,.
\end{aligned} \tag{3.46}$$

It seems very messy, but after moving to an appropriate polar coordinate system (see section 3.1 of [85]), this background (3.46) is found to be the well-known Melvin background [87–89]. In [85], it was reproduced as a Yang-Baxter deformation with the classical r-matrix (3.45). For later convenience, we will keep the expression in (3.46).

The dual parameterization of this background is given by

$$\mathrm{d}s^2_{\text{dual}} = -(\mathrm{d}x^0)^2 + \sum_{i=1}^{9}(\mathrm{d}x^i)^2\,,\quad \beta = \eta\,\bigl(x^2\,\partial_1\wedge\partial_3 - x^3\,\partial_1\wedge\partial_2\bigr)\,,\quad \tilde{\phi}=0\,. \tag{3.47}$$

Hence, under a shift $x^2\to x^2+\eta^{-1}$, the background receives the β-transformation,

$$\beta \quad\to\quad \beta + \partial_1\wedge\partial_3\,. \tag{3.48}$$

Therefore, if the x^2 direction is compactified with the period η^{-1}, then the monodromy matrix becomes

$$\mathcal{H}_{MN}(x^2+\eta^{-1}) = \bigl[\Omega^{\mathrm{T}}\mathcal{H}(x^2)\,\Omega\bigr]_{MN}\,,\qquad \Omega^M{}_N \equiv \begin{pmatrix} \delta^m_n & 2\,\delta^{[m}_1\,\delta^{n]}_3 \\ 0 & \delta^n_m \end{pmatrix} \in \mathrm{O}(10,10;\mathbb{Z})\,. \tag{3.49}$$

Thus this background has been shown to be a T-fold.

When the x^3 direction is also identified with the period η^{-1}, the corresponding monodromy matrix becomes

$$\mathcal{H}_{MN}(x^3+\eta^{-1}) = \bigl[\Omega^{\mathrm{T}}\mathcal{H}(x^3)\,\Omega\bigr]_{MN}\,,\qquad \Omega^M{}_N \equiv \begin{pmatrix} \delta^m_n & -2\,\delta^{[m}_1\,\delta^{n]}_2 \\ 0 & \delta^n_m \end{pmatrix} \in \mathrm{O}(10,10;\mathbb{Z})\,. \tag{3.50}$$

In terms of non-geometric fluxes, this background has a constant Q-flux. In the examples of T-folds presented in section 2, a background with a constant Q-flux, $Q_p{}^{mn}$, is mapped to another background with a constant H-flux, H_{pmn}, under a double T-duality along x^m and x^n directions. On the other hand, in the present example, the background has two types of constant Q-fluxes, $Q_2{}^{13}$ and $Q_3{}^{12}$, but we cannot perform T-dualities to make the background a constant-H-flux background because x^2 and x^3 directions are not isometry directions.

3.4.2 Non-unimodular example 1: $r = \frac{1}{2}(P_0 - P_1) \wedge M_{01}$

Let us consider a non-unimodular classical r-matrix[5]

$$r = \frac{1}{2}(P_0 - P_1) \wedge M_{01}\,. \tag{3.51}$$

The corresponding YB-deformed background becomes

$$\begin{aligned} \mathrm{d}s^2 &= \frac{-(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2}{1-\eta^2\,(x^0+x^1)^2} + \sum_{i=2}^{9}(\mathrm{d}x^i)^2\,, \\ B_2 &= -\frac{\eta\,(x^0+x^1)}{1-\eta^2\,(x^0+x^1)^2}\,\mathrm{d}x^0\wedge\mathrm{d}x^1\,, \qquad \Phi = \frac{1}{2}\ln\biggl[\frac{1}{1-\eta^2\,(x^0+x^1)^2}\biggr]\,. \end{aligned} \tag{3.52}$$

Apparently, this background has a coordinate singularity at $x^0 + x^1 = \pm 1/\eta$. But when the dual parameterization (2.20) is employed, the dual fields are given by

$$\mathrm{d}s^2_{\text{dual}} = -(\mathrm{d}x^0)^2 + \sum_{i=1}^{9}(\mathrm{d}x^i)^2\,, \quad \beta = \eta\,(x^0+x^1)\,\partial_0\wedge\partial_1\,, \quad \tilde{\phi} = 0\,, \tag{3.53}$$

and they are regular everywhere.[6]

By introducing a Killing vector I with the help of the divergence formula (3.11) as

$$I = \tilde{D}_n\beta^{mn}\,\partial_m = \partial_n\beta^{mn}\,\partial_m = \eta\,(\partial_0 - \partial_1)\,, \tag{3.54}$$

the background (3.52) with this I solves GSE.

Since the β-field depends on x^1 linearly, as one moves along the x^1 direction, the background is twisted by the β-transformation. In particular, when the x^1 direction is identified with period $1/\eta$, this background becomes a T-fold with an $\mathrm{O}(10,10;\mathbb{Z})$ monodromy,

$$\mathcal{H}_{MN}(x^1+\eta^{-1}) = \bigl[\Omega^{\mathrm{T}}\mathcal{H}(x^1)\,\Omega\bigr]_{MN}\,, \qquad \Omega^M{}_N \equiv \begin{pmatrix} \delta^m_n & 2\,\delta_0^{[m}\,\delta_1^{n]} \\ 0 & \delta_m^n \end{pmatrix}. \tag{3.55}$$

Note that an arbitrary solution of GSE can be regarded as a solution of DFT [32]. Indeed, by introducing the light-cone coordinates and a rescaled deformation parameter as

$$x^\pm \equiv \frac{x^0 \pm x^1}{\sqrt{2}}\,, \qquad \bar{\eta} = \sqrt{2}\,\eta\,, \tag{3.56}$$

the present YB-deformed background can be regarded as the following solution of DFT:

$$\mathcal{H} = \begin{pmatrix} 0 & -1 & -\bar{\eta}\,x^+ & 0 \\ -1 & 0 & 0 & \bar{\eta}\,x^+ \\ -\bar{\eta}\,x^+ & 0 & 0 & (\bar{\eta}\,x^+)^2 - 1 \\ 0 & \bar{\eta}\,x^+ & (\bar{\eta}\,x^+)^2 - 1 & 0 \end{pmatrix}, \qquad d = \bar{\eta}\,\tilde{x}_-\,, \tag{3.57}$$

[5] As far as we know, this example has not been discussed anywhere so far.

[6] A similar resolution of singularities in the dual parameterization has been argued in [90, 91] in the context of the exceptional field theory.

where only $(x^+, x^-, \tilde{x}_+, \tilde{x}_-)$-components of $\mathcal{H}_{MN}$ are displayed. Note here that the dilaton has an explicit dual-coordinate dependence because we are now considering a non-standard solution of the section condition which makes this background a solution of GSE rather than the usual supergravity.

Before perfoming this YB deformation (i.e. $\bar{\eta} = 0$), there is a Killing vector $\chi \equiv \partial_+$, but the associated isometry is broken for non-zero $\bar{\eta}$. However, even after deforming the geometry, there exists a generalized Killing vector

$$\chi \equiv \mathrm{e}^{\bar{\eta}\,\tilde{x}_-}\,\partial_+ \qquad \big(\hat{\mathcal{L}}_\chi \mathcal{H}_{MN} = 0\,, \quad \hat{\mathcal{L}}_\chi d = 0\big)\,, \tag{3.58}$$

which goes back to the original Killing vector in the undeformed limit, $\bar{\eta} \to 0$. In order to make the generalized isometry manifest, let us consider a generalized coordinate transformation,

$$x'^+ = \mathrm{e}^{-\bar{\eta}\,\tilde{x}_-}\,x^+\,, \qquad \tilde{x}'_- = -\bar{\eta}^{-1}\,\mathrm{e}^{-\bar{\eta}\,\tilde{x}_-}\,, \qquad x'^M = x^M \quad \text{(others)}\,. \tag{3.59}$$

By employing Hohm and Zwiebach's finite transformation matrix [92],

$$\mathcal{F}_M{}^N \equiv \frac{1}{2}\left(\frac{\partial x^K}{\partial x'^M}\frac{\partial x'_K}{\partial x_N} + \frac{\partial x'_M}{\partial x_K}\frac{\partial x^N}{\partial x'^K}\right), \tag{3.60}$$

the generalized Killing vector in the primed coordinates becomes constant, $\chi = \partial'_+$. We can also check that the generalized metric in the primed coordinate system is precisely the undeformed background. Namely, at least locally, the YB deformation can be undone by the generalized coordinate transformation.[7] This fact is consistent with the fact that YB deformations can be realized as the generalized diffeomorphism [33].

Non-Riemannian background. Since the above background has a linear coordinate dependence on $\tilde{x}_-$, let us rotate the solution to the canonical section (i.e. a section in which all of the fields are independent of the dual coordinates). By performing a T-duality along the x^- direction, we obtain

$$\mathcal{H} = \begin{pmatrix} 0 & 0 & -\bar{\eta}\,x^+ & -1 \\ 0 & 0 & (\bar{\eta}\,x^+)^2 - 1 & \bar{\eta}\,x^+ \\ -\bar{\eta}\,x^+ & (\bar{\eta}\,x^+)^2 - 1 & 0 & 0 \\ -1 & \bar{\eta}\,x^+ & 0 & 0 \end{pmatrix}, \qquad d = \bar{\eta}\,x^-\,. \tag{3.61}$$

The resulting background is indeed a solution of DFT defined on the canonical section. However, this solution cannot be parameterized in terms of $(g_{mn},\, B_{mn})$ and is called a non-Riemannian background in the terminology of [93]. This background does not even allow the dual parameterization (2.20) in terms of $(G_{mn},\, \beta^{mn})$.[8]

[7] In the study of YB deformations of AdS_5, the similar phenomenon has already been observed in [54].

[8] For another example of non-Riemannian backgrounds, see [93]. A classification of non-Riemannian backgrounds in DFT has been made in [94]. In the context of the exceptional field theory, non-Riemannian backgrounds have been found in [91] even before [93]. There, the type IV generalized metrics do not allow both the conventional and dual parameterizations similar to our solution (3.61).

3.4.3 Non-unimodular example 2: $r = \frac{1}{2\sqrt{2}} \sum_{\mu=0}^{4} (M_{0\mu} - M_{1\mu}) \wedge P^{\mu}$

The next example is the classical r-matrix [86],

$$r = \frac{1}{2\sqrt{2}} \sum_{\mu=0}^{4} \bigl(M_{0\mu} - M_{1\mu}\bigr) \wedge P^{\mu} \,. \tag{3.62}$$

This classical r-matrix is a higher dimensional generalization of the light-cone κ-Poincaré r-matrix in the four dimensional one.

By using the light-cone coordinates,

$$x^{\pm} \equiv \frac{x^0 \pm x^1}{\sqrt{2}} \,, \tag{3.63}$$

the corresponding YB-deformed background becomes

$$\begin{aligned} \mathrm{d}s^2 &= \frac{-2\,\mathrm{d}x^+\,\mathrm{d}x^- - \eta^2\,\mathrm{d}x^+\bigl[\sum_{i=2}^4 (x^i)^2\,\mathrm{d}x^+ - 2\,x^+ \sum_{i=2}^4 x^i\,\mathrm{d}x^i\bigr]}{1-(\eta\,x^+)^2} + \sum_{i=2}^{9} (\mathrm{d}x^i)^2 \,, \\ B_2 &= \frac{\eta\,\mathrm{d}x^+ \wedge \bigl(x^+\,\mathrm{d}x^- - \sum_{i=2}^4 x^i\,\mathrm{d}x^i\bigr)}{1-(\eta\,x^+)^2} \,, \qquad \Phi = \frac{1}{2}\ln\biggl[\frac{1}{1-(\eta\,x^+)^2}\biggr] \,. \end{aligned} \tag{3.64}$$

In terms of the dual parameterization, this background becomes

$$\begin{aligned} \mathrm{d}s^2_{\text{dual}} &= -2\,\mathrm{d}x^+\,\mathrm{d}x^- + \sum_{i=2}^{9} (\mathrm{d}x^i)^2 \,, \quad \tilde{\phi} = 0 \,, \\ \beta &= \eta \sum_{\mu=0}^{4} \hat{M}_{-\mu} \wedge \hat{P}^{\mu} = \eta\,\partial_- \wedge \Bigl(x^+\,\partial_+ + \sum\nolimits_{i=2}^4 x^i\,\partial_i\Bigr) \,. \end{aligned} \tag{3.65}$$

Again, by introducing a Killing vector from the divergence formula (3.11) as

$$I = 4\,\eta\,\partial_- \,, \tag{3.66}$$

the background (3.64) with this I solves GSE.

This background can also be regarded as the following solution of DFT:

$$\begin{aligned} \mathcal{H} &= \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & -\eta\,x^+ & 0 & -\eta\,x^2 & -\eta\,x^3 & -\eta\,x^4 \\ -1 & 0 & 0 & 0 & 0 & 0 & \eta\,x^+ & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\eta\,x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -\eta\,x^3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -\eta\,x^4 & 0 & 0 & 0 \\ -\eta\,x^+ & 0 & 0 & 0 & 0 & 0 & (\eta\,x^+)^2-1 & 0 & 0 & 0 \\ 0 & \eta\,x^+ & -\eta\,x^2 & -\eta\,x^3 & -\eta\,x^4 & (\eta\,x^+)^2-1 & \eta^2 \sum_{i=2}^4 (x^i)^2 & \eta^2\,x^+\,x^2 & \eta^2\,x^+\,x^3 & \eta^2\,x^+\,x^4 \\ -\eta\,x^2 & 0 & 0 & 0 & 0 & 0 & \eta^2\,x^+\,x^2 & 1 & 0 & 0 \\ -\eta\,x^3 & 0 & 0 & 0 & 0 & 0 & \eta^2\,x^+\,x^3 & 0 & 1 & 0 \\ -\eta\,x^4 & 0 & 0 & 0 & 0 & 0 & \eta^2\,x^+\,x^4 & 0 & 0 & 1 \end{pmatrix} , \\ d &= 4\,\eta\,\tilde{x}_- \,, \end{aligned} \tag{3.67}$$

where only $(x^+, x^-, x^2, x^3, x^4, \tilde{x}_+, \tilde{x}_-, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4)$-components of $\mathcal{H}_{MN}$ are displayed.

When one of the (x^2, x^3, x^4)-coordinates, say x^2, is compactified with the period $x^2 \sim x^2 + \eta^{-1}$, the monodromy matrix is given by

$$\mathcal{H}_{MN}(x^2+\eta^{-1}) = \bigl[\Omega^{\mathrm{T}}\mathcal{H}(x^2)\,\Omega\bigr]_{MN} \,, \qquad \Omega^{M}{}_{N} \equiv \begin{pmatrix} \delta_n^m & 2\,\delta_-^{[m}\,\delta_2^{n]} \\ 0 & \delta^n_m \end{pmatrix} \in \mathrm{O}(10,10;\mathbb{Z}) \,, \tag{3.68}$$

and in this sense the compactified background is a T-fold. In terms of the non-geometric Q-flux, this background has the following components of it:

$$Q_+{}^{-+} = Q_2{}^{-2} = Q_3{}^{-3} = Q_4{}^{-4} = \eta\,. \tag{3.69}$$

3.5 A non-geometric background from non-Abelian T-duality

Before considering YB-deformations of $\mathrm{AdS}_5\times\mathrm{S}^5$, let us consider another example of purely NS-NS background, which was found in [16] via a non-Abelian T-duality.

The background takes the form,

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}t^2 + \frac{(t^4+y^2)\,\mathrm{d}x^2 - 2\,x\,y\,\mathrm{d}x\,\mathrm{d}y + (t^4+x^2)\,\mathrm{d}y^2 + t^4\,\mathrm{d}z^2}{t^2\,(t^4+x^2+y^2)} + \mathrm{d}s^2_{T^6}\,, \\ B_2 &= \frac{(x\,\mathrm{d}x + y\,\mathrm{d}y)\wedge \mathrm{d}z}{t^4+x^2+y^2}\,, \qquad \Phi = \frac{1}{2}\ln\left[\frac{1}{t^2\,(t^4+x^2+y^2)}\right], \end{aligned} \tag{3.70}$$

where $\mathrm{d}s^2_{T^6}$ is the flat metric on a 6-torus. In terms of the dual parameterization, this background takes a Friedmann-Robertson-Walker-type form,

$$\begin{aligned} \mathrm{d}s^2_{\mathrm{dual}} &= -\mathrm{d}t^2 + t^{-2}\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right) + \mathrm{d}s^2_{T^6}\,, \\ \beta &= (x\,\partial_x + y\,\partial_y)\wedge\partial_z\,, \qquad \tilde{\phi} = -\ln t^3\,. \end{aligned} \tag{3.71}$$

Note here that this background cannot be represented by a coset or a Lie group itself. This is because the background (3.70) contains a curvature singularity and is not homogeneous. Hence the background (3.70) cannot be realized as a Yang-Baxter deformation and is not included in the discussion of [27–29].

It is easy to see that the associated Q-flux is constant on this background (3.71),

$$Q_y{}^{xy} = Q_z{}^{xz} = -1\,. \tag{3.72}$$

Therefore, if the x-direction is compactified as $x\sim x+1$, the background fields are twisted by an $\mathrm{O}(10,10;\mathbb{Z})$ transformation as

$$\mathcal{H}_{MN}(x+1) = \left[\Omega^{\mathrm{T}}\mathcal{H}(x)\,\Omega\right]_{MN}, \qquad \Omega^M{}_N \equiv \begin{pmatrix} \delta^m_n & 2\,\delta^{[m}_x\,\delta^{n]}_z \\ 0 & \delta^n_m \end{pmatrix}, \qquad d(x+1) = d(x)\,. \tag{3.73}$$

Thus the background can be interpreted as a T-fold. If the z-direction is also compactified as $z\sim z+1$, another twist is realized as

$$\mathcal{H}_{MN}(y+1) = \left[\Omega^{\mathrm{T}}\mathcal{H}(y)\,\Omega\right]_{MN}, \qquad \Omega^M{}_N \equiv \begin{pmatrix} \delta^m_n & 2\,\delta^{[m}_y\,\delta^{n]}_z \\ 0 & \delta^n_m \end{pmatrix}, \qquad d(y+1) = d(y)\,. \tag{3.74}$$

As stated in [16], this background is not a solution of the usual supergravity. However, by using the divergence formula $I^m = \tilde{D}_n\beta^{mn}$ again and introducing a vector field as

$$I = -2\,\partial_z\,, \tag{3.75}$$

we can see that the background (3.70) together with this vector field I satisfies GSE. Thus, this background can also be regarded as a T-fold solution of DFT.

In this paper, we have considered just one example of non-Abelian T-duality, but it would be interesting to study a lot of examples as a new technique to generate GSE solutions. In fact, it is well-known that non-Abelian T-duality is a systematic method to construct T-fold solutions in DFT.

JHEP12(2017)108

3.6 YB-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ backgrounds

We show that various YB deformations of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background are T-folds. We consider here examples associated with the following five classical r-matrices:

1. $r = \frac{1}{2\eta}\left[\eta_1\,(D + M_{+-}) \wedge P_+ + \eta_2\, M_{+2} \wedge P_3\right]$,
2. $r = \frac{1}{2}\, P_0 \wedge D$,
3. $r = \frac{1}{2}\left[P_0 \wedge D + P^i \wedge (M_{0i} + M_{1i})\right]$,
4. $r = \frac{1}{2\eta}\, P_- \wedge (\eta_1\, D - \eta_2\, M_{+-})$,
5. $r = \frac{1}{2}\, M_{-\mu} \wedge P^\mu$.

The classical r-matrices other than the first one are non-unimodular. Note here that the S^5 part remains undeformed and only the AdS_5 part is deformed. As shown in appendix A, through the (modified) Penrose limit, the second and third examples are reduced to the two examples discussed in the previous subsection.

3.6.1 Non-Abelian unimodular r-matrix

Let us consider a non-Abelian unimodular r-matrix (see R_5 in table 1 of [55]),

$$r = \frac{1}{2\,\eta}\left[\eta_1\,(D + M_{+-}) \wedge P_+ + \eta_2\, M_{+2} \wedge P_3\right], \tag{3.76}$$

where, for simplicity, it is written in terms of the light-cone coordinates,[9]

$$x^\pm \equiv \frac{x^0 \pm x^1}{\sqrt{2}}\,. \tag{3.77}$$

The corresponding YB-deformed background is given by

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{\mathrm{d}z^2}{z^2} + \frac{z^2\,[(\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2]}{z^4 + (\eta_2\, x^-)^2} - \frac{2\,z^2\,\mathrm{d}x^+\,\mathrm{d}x^- - 4\eta_1^2\, z^{-1}\, x^-\,\mathrm{d}z\,\mathrm{d}x^-}{z^4 - (2\eta_1\, x^-)^2} \\
&\quad + \frac{2\{[x^2(2\eta_1^2+\eta_2^2) - \eta_1\,\eta_2\, x^3]\, z^2\, x^-\,\mathrm{d}x^2 + \eta_1\,(2\eta_1\, x^3 - \eta_2\, x^2)\,\mathrm{d}x^3\}\,\mathrm{d}x^-}{[z^4 - (2\eta_1\, x^-)^2]\,[z^4 + (\eta_2\, x^-)^2]} \\
&\quad - \frac{(\eta_1^2+\eta_2^2)\,(z\,x^2)^2 - 2\eta_1\,\eta_2\, z^2\, x^2\, x^3 + \eta_1^2\,[z^4 + (z\,x^3)^2 + (\eta_2\, x^-)^2]}{[z^4 - (2\eta_1\, x^-)^2]\,[z^4 + (\eta_2\, x^-)^2]}\,(\mathrm{d}x^-)^2 + \mathrm{d}s^2_{\mathrm{S}^5}\,, \\
B_2 &= -\left[\frac{\eta_1\,\{x^2\,[z^4 + 2(\eta_2\, x^-)^2] - 2\eta_1\,\eta_2\,(x^-)^2\, x^3\}\,\mathrm{d}x^2 + \{\eta_1\, z^4\, x^3 - \eta_2\, x^2\,[z^4 - 2(\eta_1\, x^-)^2]\}\,\mathrm{d}x^3}{[z^4 - (2\eta_1\, x^-)^2]\,[z^4 + (\eta_2\, x^-)^2]}\right. \\
&\quad \left. + \frac{\eta_1\,(z\,\mathrm{d}z - 2x^-\,\mathrm{d}x^+)}{z^4 - (2\eta_1\, x^-)^2}\right] \wedge \mathrm{d}x^- + \frac{\eta_2\, x^-\,\mathrm{d}x^2 \wedge \mathrm{d}x^3}{z^4 + (\eta_2\, x^-)^2}\,, \\
\Phi &= \frac{1}{2}\ln\left[\frac{z^8}{[z^4 - (2\eta_1\, x^-)^2]\,[z^4 + (\eta_2\, x^-)^2]}\right], \\
\hat{\mathsf{F}}_1 &= \frac{4\eta_1\,\eta_2\, x^-\,(2x^-\,\mathrm{d}z - z\,\mathrm{d}x^-)}{z^5}\,,
\end{aligned}$$

[9] In the following, our light-cone convention is taken as $\varepsilon_{z+-23r\xi\phi_1\phi_2\phi_3} = +\sqrt{|g|}$ rather than (3.28).

$$
\begin{aligned}
\hat{\mathsf{F}}_3 &= -B_2\wedge\mathsf{F}_1 + \frac{4\eta_1}{z^5}\left(2x^-\mathrm{d}z - z\,\mathrm{d}x^-\right)\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3 \\
&\quad + \frac{4}{z^5}\,\mathrm{d}z\wedge\mathrm{d}x^-\wedge\left[\eta_1\,(x^3\mathrm{d}x^2 - x^2\mathrm{d}x^3) + \eta_2\,(x^-\mathrm{d}x^+ - x^2\mathrm{d}x^2)\right], \\
\hat{\mathsf{F}}_5 &= 4\left[\frac{z^8}{[z^4-(2\eta_1 x^-)^2][z^4+(\eta_2 x^-)^2]}\,\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}\right], \\
\hat{\mathsf{F}}_7 &= -B_2\wedge\mathsf{F}_5\,, \qquad \hat{\mathsf{F}}_9 = -\frac{1}{2}\,B_2\wedge\mathsf{F}_7\,.
\end{aligned}
\tag{3.78}
$$

In terms of the dual fields, we obtain the following expression:

$$
\begin{gathered}
\mathrm{d}s^2_{\text{dual}} = \frac{\mathrm{d}z^2 - 2\,\mathrm{d}x^+\,\mathrm{d}x^- + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2}{z^2} + \mathrm{d}s^2_{\mathrm{S}^5}\,, \qquad \tilde{\phi} = 0\,, \\
\beta = \eta_1\left(z\,\partial_z + 2\,x^-\,\partial_- + x^2\,\partial_2 + x^3\,\partial_3\right)\wedge\partial_+ + \eta_2\left(x^2\,\partial_+ + x^-\,\partial_2\right)\wedge\partial_3\,.
\end{gathered}
\tag{3.79}
$$

It is straightforward to check that the R-R field strengths are given by

$$
\hat{\mathsf{F}} = \mathrm{e}^{-B_2\wedge}\,\mathsf{F}\,, \qquad \mathsf{F} \equiv \mathrm{e}^{-\beta\vee}\,\check{\mathsf{F}}\,, \qquad \check{\mathsf{F}} = 4\left(\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}\right). \tag{3.80}
$$

Namely, as advocated in section 3.2, the β-untwisted R-R fields $\check{\mathsf{F}}$ are invariant under the YB deformation.

This background has the following components of Q-flux:

$$
Q_z{}^{z+} = \eta_1\,, \quad Q_-{}^{-+} = 2\,\eta_1\,, \quad Q_2{}^{2+} = \eta_1\,, \quad Q_3{}^{3+} = \eta_1\,, \quad Q_2{}^{+3} = \eta_2\,, \quad Q_-{}^{23} = \eta_2\,. \tag{3.81}
$$

Accordingly, for example, when the x^3 direction is compactified with a period $x^3 \sim x^3+\eta_1^{-1}$, this background becomes a T-fold with the monodromy,

$$
\mathcal{H}_{MN}(x^3+\eta_1^{-1}) = \left[\Omega^{\mathrm{T}}\mathcal{H}(x)\,\Omega\right]_{MN}\,, \qquad \Omega^M{}_N \equiv \begin{pmatrix} \delta^m_n & 2\,\delta_3^{[m}\,\delta_+^{n]} \\ 0 & \delta_m^n \end{pmatrix} \in \mathrm{O}(10,10;\mathbb{Z})\,. \tag{3.82}
$$

The R-R fields F are also twisted by the same monodromy,

$$
\mathsf{F}(x^3+\eta_1^{-1}) = \mathrm{e}^{-\omega\vee}\,\mathsf{F}(x^3)\,, \qquad \omega^{mn} = 2\,\delta_3^{[m}\,\delta_+^{n]}\,. \tag{3.83}
$$

Note that the R-R potentials are twisted by the same monodromy as well, though their explicit forms are not written down here.

3.6.2 $r = \frac{1}{2}\,P_0\wedge D$

Let us next consider a classical r-matrix [50, 54],

$$
r = \frac{1}{2}\,P_0\wedge D\,. \tag{3.84}
$$

Because $[P_0, D]\neq 0$, this classical r-matrix does not satisfy the unimodularity condition. By introducing the polar coordinates,

$$
x^1 = \rho\sin\theta\cos\phi\,, \qquad x^2 = \rho\sin\theta\sin\phi\,, \qquad x^3 = \rho\cos\theta\,, \tag{3.85}
$$

the deformed background can be rewritten as [54][10]

$$
\begin{aligned}
\mathrm{d}s^2 &= \frac{z^2\left[\mathrm{d}z^2-(\mathrm{d}x^0)^2+\mathrm{d}\rho^2\right]-\eta^2\,(\mathrm{d}\rho-\rho\,z^{-1}\,\mathrm{d}z)^2}{z^4-\eta^2\,(z^2+\rho^2)}+\frac{\rho^2\,(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)}{z^2}+\mathrm{d}s^2_{\mathrm{S}^5}\,,\\
B_2 &= -\eta\,\frac{\mathrm{d}x^0\wedge(z\,\mathrm{d}z+\rho\,\mathrm{d}\rho)}{z^4-\eta^2\,(z^2+\rho^2)}\,,\qquad \Phi=\frac{1}{2}\ln\left[\frac{z^4}{z^4-\eta^2\,(z^2+\rho^2)}\right],\qquad I=-\eta\,\partial_0\,,\\
\hat{\mathsf{F}}_1 &= 0\,,\quad \hat{\mathsf{F}}_3=\frac{4\,\eta\,\rho^2\sin\theta}{z^5}\,(z\,\mathrm{d}\rho-\rho\,\mathrm{d}z)\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi\,,\\
\hat{\mathsf{F}}_5 &= 4\left[\frac{z^4}{z^4-\eta^2\,(z^2+\rho^2)}\,\omega_{\mathrm{AdS}_5}+\omega_{\mathrm{S}^5}\right],\\
\hat{\mathsf{F}}_7 &= \frac{4\,\eta\,\mathrm{d}x^0\wedge(z\,\mathrm{d}z+\rho\,\mathrm{d}\rho)}{z^4-\eta^2\,(z^2+\rho^2)}\wedge\omega_{\mathrm{S}^5}\,,\quad \hat{\mathsf{F}}_9=0\,.
\end{aligned}
\tag{3.86}
$$

This background is not a solution of the usual type IIB supergravity, but that of GSE [64]. By setting $\eta=0$, this background reduces to the original $\mathrm{AdS}_5\times\mathrm{S}^5$.

In the dual parameterization, the dual metric, the β field and the dual dilaton are given by

$$
\begin{aligned}
&\mathrm{d}s^2_{\text{dual}}=\frac{\mathrm{d}z^2-(\mathrm{d}x^0)^2+(\mathrm{d}x^1)^2+(\mathrm{d}x^2)^2+(\mathrm{d}x^3)^2}{z^2}+\mathrm{d}s^2_{\mathrm{S}^5}\,,\qquad \tilde{\phi}=0\,,\\
&\beta=\eta\,\hat{P}_0\wedge\hat{D}=\eta\,\partial_0\wedge(z\,\partial_z+x^1\,\partial_1+x^2\,\partial_2+x^3\,\partial_3)=\eta\,\partial_0\wedge(z\,\partial_z+\rho\,\partial_\rho)\,.
\end{aligned}
\tag{3.87}
$$

The Killing vector I^m satisfies the divergence formula,

$$
I^0=-\eta=\tilde{D}_m\beta^{0m}\,. \tag{3.88}
$$

The Q-flux has the following non-vanishing components:

$$
Q_z{}^{0z}=Q_1{}^{01}=Q_2{}^{02}=Q_3{}^{03}=\eta\,. \tag{3.89}
$$

Thus, when at least one of the (x^1,x^2,x^3) directions is compactified, the background can be interpreted as a T-fold. For example, when the x^1 direction is compactified, the monodromy is given by

$$
\mathcal{H}_{MN}(x^1+\eta^{-1})=\left[\Omega^{\mathrm{T}}\mathcal{H}(x^1)\,\Omega\right]_{MN}\,,\qquad \Omega^M{}_N\equiv\begin{pmatrix}\delta^m_n & 2\,\delta_0^{[m}\,\delta_1^{n]}\\ 0 & \delta^n_m\end{pmatrix}. \tag{3.90}
$$

From (3.86), the R-R potentials can be found as follows:

$$
\begin{aligned}
&\hat{\mathsf{C}}_0=0\,, && \hat{\mathsf{C}}_2=\frac{\eta\,\rho^3\sin\theta}{z^4}\,\mathrm{d}\theta\wedge\mathrm{d}\phi\,,\\
&\hat{\mathsf{C}}_4=\frac{\rho^2\sin\theta}{z^4}\,\mathrm{d}x^0\wedge\mathrm{d}\rho\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi+\omega_4-B_2\wedge\hat{\mathsf{C}}_2\,,\\
&\hat{\mathsf{C}}_6=-B_2\wedge\omega_4\,, && \hat{\mathsf{C}}_8=0\,.
\end{aligned}
\tag{3.91}
$$

[10] Only the metric and NS-NS two-form were computed in [50].

JHEP12(2017)108

Providing the B-twist, we obtain

$$\begin{aligned}
&\mathsf{F}_1 = 0\,, && \mathsf{F}_3 = \frac{4\,\eta\,\rho^2\sin\theta}{z^5}\,(\rho\,\mathrm{d}z - z\,\mathrm{d}\rho)\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi\,, \\
&\mathsf{F}_5 = 4\,\bigl(\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}\bigr)\,, && \mathsf{F}_7 = 0\,, \qquad \mathsf{F}_9 = 0\,, \\
&\mathsf{A}_0 = 0\,, && \mathsf{A}_2 = \frac{\eta\,\rho^3\sin\theta}{z^4}\,\mathrm{d}\theta\wedge\mathrm{d}\phi\,, \\
&\mathsf{A}_4 = \frac{\rho^2\sin\theta}{z^4}\,\mathrm{d}x^0\wedge\mathrm{d}\rho\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi + \omega_4\,, && \mathsf{A}_6 = 0\,, \qquad \mathsf{A}_8 = 0\,.
\end{aligned} \tag{3.92}$$

We can further compute the β-untwisted fields,

$$\begin{aligned}
&\check{\mathsf{F}}_1 = 0\,, \quad \check{\mathsf{F}}_3 = 0\,, \quad \check{\mathsf{F}}_5 = 4\,\bigl(\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}\bigr)\,, && \check{\mathsf{F}}_7 = 0\,, \quad \check{\mathsf{F}}_9 = 0\,, \\
&\check{\mathsf{C}}_0 = 0\,, \quad \check{\mathsf{C}}_2 = 0\,, \quad \check{\mathsf{C}}_4 = \frac{\rho^2\sin\theta}{z^4}\,\mathrm{d}x^0\wedge\mathrm{d}\rho\wedge\mathrm{d}\theta\wedge\mathrm{d}\phi + \omega_4\,, && \check{\mathsf{C}}_6 = 0\,, \quad \check{\mathsf{C}}_8 = 0\,.
\end{aligned} \tag{3.93}$$

As expected, the β-untwisted R-R fields are precisely the R-R fields in the undeformed background, and they are single-valued. In terms of the twisted R-R fields, $(\mathsf{F},\,\mathsf{A})$, the R-R fields have the same monodromy as (3.90),

$$\mathsf{A}(x^1+\eta^{-1}) = \mathrm{e}^{-\omega\vee}\,\mathsf{A}(x^1)\,, \quad \mathsf{F}(x^1+\eta^{-1}) = \mathrm{e}^{-\omega\vee}\,\mathsf{F}(x^1)\,, \quad \omega^{mn} = 2\,\delta_0^{[m}\,\delta_1^{n]}\,. \tag{3.94}$$

3.6.3 A scaling limit of the Drinfeld-Jimbo r-matrix

Let us consider a classical r-matrix [53, 54],

$$r = \frac{1}{2}\,\bigl[P_0\wedge D + P^i\wedge(M_{0i}+M_{1i})\bigr]\,, \tag{3.95}$$

which can be obtained as a scaling limit of the classical r-matrix of Drinfeld-Jimbo type [39, 40]. By using the polar coordinates (ρ,θ),

$$(\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 = \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2\,, \tag{3.96}$$

the YB-deformed background, which satisfies GSE, is given by [53, 54]

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{\mathrm{d}z^2 - (\mathrm{d}x^0)^2}{z^2-\eta^2} + \frac{z^2\bigl[(\mathrm{d}x^1)^2 + \mathrm{d}\rho^2\bigr]}{z^4+\eta^2\,\rho^2} + \frac{\rho^2\,\mathrm{d}\theta^2}{z^2} + \mathrm{d}s^2_{\mathrm{S}^5}\,, \\
B_2 &= \eta\left[\frac{\mathrm{d}z\wedge\mathrm{d}x^0}{z\,(z^2-\eta^2)} - \frac{\rho\,\mathrm{d}x^1\wedge\mathrm{d}\rho}{z^4+\eta^2\,\rho^2}\right], \\
\Phi &= \frac{1}{2}\ln\!\left[\frac{z^6}{(z^2-\eta^2)(z^4+\eta^2\,\rho^2)}\right], \qquad I = -\eta\,(4\,\partial_0 + 2\,\partial_1)\,, \\
\hat{\mathsf{F}}_1 &= -\frac{4\,\eta^2\,\rho^2}{z^4}\,\mathrm{d}\theta\,, \quad \hat{\mathsf{F}}_3 = 4\,\eta\,\rho\left(\frac{\rho\,\mathrm{d}z\wedge\mathrm{d}x^0}{z\,(z^4-\eta^2\,z^2)} + \frac{\mathrm{d}x^1\wedge\mathrm{d}\rho}{z^4+\eta^2\,\rho^2}\right)\wedge\mathrm{d}\theta\,, \\
\hat{\mathsf{F}}_5 &= 4\left[\frac{z^6}{(z^2-\eta^2)(z^4+\eta^2\,\rho^2)}\,\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}\right], \\
\hat{\mathsf{F}}_7 &= 4\,\eta\left(-\frac{\mathrm{d}z\wedge\mathrm{d}x^0}{z\,(z^2-\eta^2)} + \frac{\rho\,\mathrm{d}x^1\wedge\mathrm{d}\rho}{z^4+\eta^2\,\rho^2}\right)\wedge\omega_{\mathrm{S}^5}\,, \\
\hat{\mathsf{F}}_9 &= -\frac{4\,\eta^2\,\rho}{z\,(z^2-\eta^2)(z^4+\eta^2\,\rho^2)}\,\mathrm{d}z\wedge\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}\rho\wedge\omega_{\mathrm{S}^5}\,.
\end{aligned} \tag{3.97}$$

The R-R potentials can be found as follows:

$$\hat{\mathsf{C}}_0 = 0\,, \qquad \hat{\mathsf{C}}_2 = -\frac{\eta\,\rho^2}{z^4}\,\mathrm{d}x^0\wedge\mathrm{d}\theta\,, \quad \hat{\mathsf{C}}_4 = \frac{\rho}{z^4+\eta^2\,\rho^2}\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}\rho\wedge\mathrm{d}\theta+\omega_4\,,$$
$$\hat{\mathsf{C}}_6 = -B_2\wedge\omega_4\,, \quad \hat{\mathsf{C}}_8 = \frac{\eta^2\,\rho}{z\,(z^2-\eta^2)(z^4+\eta^2\,\rho^2)}\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}\rho\wedge\mathrm{d}z\wedge\omega_4\,. \tag{3.98}$$

Then the corresponding dual fields in the NS-NS sector are given by

$$\begin{aligned}\mathrm{d}s^2_{\text{dual}} &= \frac{\mathrm{d}z^2-(\mathrm{d}x^0)^2+(\mathrm{d}x^1)^2+\mathrm{d}\rho^2+\rho^2\,\mathrm{d}\theta^2}{z^2}+\mathrm{d}s^2_{\mathrm{S}^5}\,, \qquad \tilde{\phi}=0\,,\\ \beta &= \eta\left[\hat{P}_0\wedge\hat{D}+\hat{P}^i\wedge(M_{0i}+M_{1i})\right] = \eta\,(z\,\partial_0\wedge\partial_z - x^2\,\partial_1\wedge\partial_2 - x^3\,\partial_1\wedge\partial_3)\\ &= \eta\,(z\,\partial_0\wedge\partial_z-\rho\,\partial_1\wedge\partial_\rho)\,,\end{aligned}$$

and the Killing vector I^m again satisfies the divergence formula,

$$I^0 = -4\,\eta = \tilde{D}_m\beta^{0m}\,, \qquad I^1 = -2\,\eta = \tilde{D}_m\beta^{1m}\,. \tag{3.99}$$

Providing the B-twist to the R-R field strengths, we obtain

$$\begin{aligned}&\mathsf{F}_1 = -\frac{4\,\eta^2\rho^2}{z^4}\,\mathrm{d}\theta\,, && \mathsf{F}_3 = \frac{4\,\eta\,\rho}{z^5}\,\big(\rho\,\mathrm{d}z\wedge\mathrm{d}x^0+z\,\mathrm{d}x^1\wedge\mathrm{d}\rho\big)\wedge\mathrm{d}\theta\,, &\\ &\mathsf{F}_5 = 4\,\big(\omega_{\mathrm{AdS}_5}+\omega_{\mathrm{S}^5}\big)\,, && \mathsf{F}_7 = 0\,, & \mathsf{F}_9 = 0\,,\\ &\mathsf{A}_0 = 0\,, && \mathsf{A}_2 = -\frac{\eta\,\rho^2}{z^4}\,\mathrm{d}x^0\wedge\mathrm{d}\theta\,, &\\ &\mathsf{A}_4 = \frac{\rho}{z^4}\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}\rho\wedge\mathrm{d}\theta+\omega_4\,, && \mathsf{A}_6 = 0\,, & \mathsf{A}_8 = 0\,.\end{aligned} \tag{3.100}$$

Furthermore, the β-untwist leads to the following expressions:

$$\begin{aligned}&\check{\mathsf{F}}_1 = 0\,, \quad \check{\mathsf{F}}_3 = 0\,, \quad \check{\mathsf{F}}_5 = 4\,\big(\omega_{\mathrm{AdS}_5}+\omega_{\mathrm{S}^5}\big)\,, && \check{\mathsf{F}}_7 = 0\,, \quad \check{\mathsf{F}}_9 = 0\,,\\ &\check{\mathsf{C}}_0 = 0\,, \quad \check{\mathsf{C}}_2 = 0\,, \quad \check{\mathsf{C}}_4 = \frac{\rho}{z^4}\,\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\mathrm{d}\rho\wedge\mathrm{d}\theta+\omega_4\,, && \check{\mathsf{C}}_6 = 0\,, \quad \check{\mathsf{C}}_8 = 0\,.\end{aligned} \tag{3.101}$$

These are the same as the undeformed R-R potentials.

Then the non-zero component of Q-flux are given by

$$Q_z{}^{0z} = \eta\,, \qquad Q_2{}^{12} = -\eta\,, \qquad Q_3{}^{13} = -\eta\,. \tag{3.102}$$

When the x^2-direction is compactified as $x^2 \sim x^2+\eta^{-1}$, this background becomes a T-fold with the monodromy,

$$\begin{aligned}\mathcal{H}_{MN}(x^2+\eta^{-1}) &= \big[\Omega^{\mathrm{T}}\mathcal{H}(x^2)\,\Omega\big]_{MN}\,, \quad \Omega^M{}_N \equiv \begin{pmatrix}\delta^m_n & -2\,\delta_1^{[m}\,\delta_2^{n]}\\ 0 & \delta_m^n\end{pmatrix}\,,\\ \mathsf{F}(x^2+\eta^{-1}) &= \mathrm{e}^{-\omega\vee}\,\mathsf{F}(x^2)\,, \qquad \omega^{mn} = -2\,\delta_1^{[m}\,\delta_2^{n]}\,.\end{aligned} \tag{3.103}$$

3.6.4 $r = \frac{1}{2\eta} P_- \wedge (\eta_1 D - \eta_2 M_{+-})$

Let us consider a non-unimodular r-matrix,[11]

$$r = \frac{1}{2\,\eta}\, P_- \wedge (\eta_1\, D - \eta_2\, M_{+-})\,. \tag{3.104}$$

Here we have introduced the light-cone coordinates and polar coordinates as

$$x^{\pm} \equiv \frac{x^0 \pm x^1}{\sqrt{2}}\,, \qquad (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 = \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2\,. \tag{3.105}$$

The YB-deformed background is given by

$$\begin{aligned}
\mathrm{d}s^2 &= \frac{\mathrm{d}z^2 + \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2}{z^2} - \frac{2\,z^2\,\mathrm{d}x^+\,\mathrm{d}x^-}{z^4 - (\eta_1+\eta_2)^2\,(x^+)^2} \\
&\quad + \eta_1\,\mathrm{d}x^+\,\frac{2\,x^+\,(\eta_1+\eta_2)\,(z\,\mathrm{d}z + \rho\,\mathrm{d}\rho) - \eta_1\,(z^2+\rho^2)\,\mathrm{d}x^+}{z^2\,[z^4 - (\eta_1+\eta_2)^2\,(x^+)^2]} + \mathrm{d}s^2_{\mathrm{S}^5}\,, \\
B_2 &= \eta_1\,\frac{x^+\,\mathrm{d}x^+\wedge\mathrm{d}x^- + z\,\mathrm{d}z\wedge\mathrm{d}x^+ - \rho\,\mathrm{d}x^+\wedge\mathrm{d}\rho}{z^4 - (\eta_1+\eta_2)^2\,(x^+)^2} + \eta_2\,\frac{x^+\,\mathrm{d}x^+\wedge\mathrm{d}x^-}{z^4 - (\eta_1+\eta_2)^2\,(x^+)^2}\,, \\
\Phi &= \frac{1}{2}\ln\biggl[\frac{z^4}{z^4 - (\eta_1+\eta_2)^2\,(x^+)^2}\biggr]\,, \qquad I = -(\eta_1-\eta_2)\,\partial_-\,, \\
\hat{\mathsf{F}}_1 &= 0\,, \quad \hat{\mathsf{F}}_3 = -\frac{4\,\rho\,\bigl[\eta_1\,(\rho\,\mathrm{d}z\wedge\mathrm{d}x^+ + z\,\mathrm{d}x^+\wedge\mathrm{d}\rho - x^+\,\mathrm{d}z\wedge\mathrm{d}\rho) - \eta_2\,x^+\,\mathrm{d}z\wedge\mathrm{d}\rho\bigr]\wedge\mathrm{d}\theta}{z^5}\,, \\
\hat{\mathsf{F}}_5 &= 4\biggl[\frac{z^4}{z^4 - (\eta_1+\eta_2)^2\,(x^+)^2}\,\omega_{\mathrm{AdS}_5} + \omega_{\mathrm{S}^5}\biggr]\,, \\
\hat{\mathsf{F}}_7 &= -\frac{4\,\bigl[\eta_1\,(x^+\,\mathrm{d}x^+\wedge\mathrm{d}x^- + z\,\mathrm{d}z\wedge\mathrm{d}x^+ - \rho\,\mathrm{d}x^+\wedge\mathrm{d}\rho) + \eta_2\,x^+\,\mathrm{d}x^+\wedge\mathrm{d}x^-\bigr]\wedge\omega_{\mathrm{S}^5}}{z^4 - (\eta_1+\eta_2)^2\,(x^+)^2}\,, \\
\hat{\mathsf{F}}_9 &= 0\,.
\end{aligned} \tag{3.106}$$

The R-R potentials are also given by

$$\begin{aligned}
&\hat{\mathsf{C}}_0 = 0\,, \qquad\qquad \hat{\mathsf{C}}_2 = \frac{\rho\,[\eta_1\,\rho\,\mathrm{d}x^+ - (\eta_1+\eta_2)\,x^+\,\mathrm{d}\rho]\wedge\mathrm{d}\theta}{z^4}\,, \\
&\hat{\mathsf{C}}_4 = \frac{\rho\,\mathrm{d}x^+\wedge[z^3\,\mathrm{d}x^- - \eta_1\,(\eta_1+\eta_2)\,x^+\,\mathrm{d}z]\wedge\mathrm{d}\rho\wedge\mathrm{d}\theta}{z^3\,[z^4 - (\eta_1+\eta_2)^2\,(x^+)^2]} + \omega_4\,, \\
&\hat{\mathsf{C}}_6 = -B_2\wedge\omega_4\,, \qquad \hat{\mathsf{C}}_8 = 0\,.
\end{aligned} \tag{3.107}$$

The dual fields are given by

$$\begin{aligned}
\mathrm{d}s^2_{\mathrm{dual}} &= \frac{\mathrm{d}z^2 - 2\,\mathrm{d}x^+\,\mathrm{d}x^- + \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2}{z^2} + \mathrm{d}s^2_{\mathrm{S}^5}\,, \qquad \tilde{\phi} = 0\,, \\
\beta &= \hat{P}_-\wedge(\eta_1\,\hat{D} + \eta_2\,\hat{M}_{+-}) = \eta_1\,\partial_-\wedge(z\,\partial_z + x^+\,\partial_+ + \rho\,\partial_\rho) + \eta_2\,x^+\,\partial_-\wedge\partial_+ \\
&= \eta_1\,\partial_-\wedge(z\,\partial_z + x^+\,\partial_+ + x^2\,\partial_2 + x^3\,\partial_3) + \eta_2\,x^+\,\partial_-\wedge\partial_+\,,
\end{aligned} \tag{3.108}$$

[11]This r-matrix includes the known examples studied in section 4.3 ($\eta_1 = -\eta_2 = -\eta$) and 4.4 ($\eta_1 = -\eta$, $\eta_2 = 0$) of [54] as special cases.

and the Q-flux has the following non-vanishing components:

$$Q_z{}^{-z} = Q_+{}^{-+} = Q_2{}^{-2} = Q_3{}^{-3} = \eta_1\,, \qquad Q_+{}^{-+} = \eta_2\,. \tag{3.109}$$

In a similar manner as the previous examples, by compactifying one of the x^1, x^2, and x^3 directions with a certain period, this background can also be regarded as a T-fold. For example, if we make the identification, $x^3 \sim x^3 + \eta_1^{-1}$, the associated monodromy becomes

$$\begin{aligned} \mathcal{H}_{MN}(x^3+\eta_1^{-1}) &= \bigl[\Omega^{\rm T}\mathcal{H}(x^3)\,\Omega\bigr]_{MN}\,, \quad \Omega^M{}_N \equiv \begin{pmatrix} \delta^m_n & 2\,\delta_-^{[m}\,\delta_3^{n]} \\ 0 & \delta^n_m \end{pmatrix}, \\ \mathsf{F}(x^3+\eta_1^{-1}) &= \mathrm{e}^{-\omega^\vee}\,\mathsf{F}(x^3)\,, \qquad\qquad \omega^{mn} = 2\,\delta_-^{[m}\,\delta_3^{n]}\,. \end{aligned} \tag{3.110}$$

A solution of generalized type IIA supergravity equations. In the background (3.97), by performing a T-duality along the x^1-direction (see [32] for the duality transformation rule), we obtain the following solution of the generalized type IIA equations of motion:

$$\begin{aligned} \mathrm{d}s^2 &= \frac{\mathrm{d}z^2-(\mathrm{d}x^0)^2}{z^2-\eta^2} + z^2\,(\mathrm{d}x^1)^2 + \frac{(\mathrm{d}\rho+\eta\,\rho\,\mathrm{d}x^1)^2+\rho^2\,\mathrm{d}\theta^2}{z^2} + \mathrm{d}s_{\mathrm{S}^5}\,, \\ B_2 &= \frac{\eta\,\mathrm{d}z\wedge\mathrm{d}x^0}{z\,(z^2-\eta^2)}\,, \qquad \Phi = -2\,\eta\,x^1 - \frac{1}{2}\ln\biggl(\frac{z^2-\eta^2}{z^4}\biggr)\,, \qquad I = -4\,\eta\,\partial_0\,, \\ \hat{\mathsf{F}}_2 &= \frac{4\,\eta\,\mathrm{e}^{2\,\eta\,x^1}\,\rho\,(\mathrm{d}\rho+\eta\,\rho\,\mathrm{d}x^1)\wedge\mathrm{d}\theta}{z^4}\,, \\ \hat{\mathsf{F}}_4 &= -\frac{4\,\mathrm{e}^{2\,\eta\,x^1}\,\rho\,\mathrm{d}z\wedge\mathrm{d}x^0\wedge(\mathrm{d}\rho+\eta\,\rho\,\mathrm{d}x^1)\wedge\mathrm{d}\theta}{z^3\,(z^2-\eta^2)}\,, \\ \hat{\mathsf{F}}_6 &= -4\,\mathrm{e}^{2\,\eta\,x^1}\,\mathrm{d}x^1\wedge\omega_{\mathrm{S}^5}\,, \qquad \hat{\mathsf{F}}_8 = \frac{4\,\eta\,\mathrm{e}^{2\,\eta\,x^1}\,\mathrm{d}z\wedge\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\omega_{\mathrm{S}^5}}{z\,(z^2-\eta^2)}\,. \end{aligned} \tag{3.111}$$

Here the R-R potentials are given by

$$\begin{aligned} \hat{\mathsf{C}}_1 &= 0\,, & \hat{\mathsf{C}}_3 &= \mathrm{e}^{2\,\eta\,x^1}\,\frac{\rho\,\mathrm{d}x^0\wedge(\mathrm{d}\rho+\eta\,\rho\,\mathrm{d}x^1)\wedge\mathrm{d}\theta}{z^4}\,, \\ \hat{\mathsf{C}}_5 &= \mathrm{e}^{2\,\eta\,x^1}\,\mathrm{d}x^1\wedge\omega_4\,, & \hat{\mathsf{C}}_7 &= -\,\mathrm{e}^{2\,\eta\,x^1}\,\frac{\eta\,\mathrm{d}z\wedge\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\omega_4}{z\,(z^2-\eta^2)}\,. \end{aligned} \tag{3.112}$$

This background cannot be regarded as a T-fold, but it is the first example of the solution for the generalized type IIA supergravity equations.

3.6.5 $r = \frac{1}{2}\,M_{-\mu}\wedge P^\mu$

The final example is associated with the r-matrix [54]

$$r = \frac{1}{2}\,M_{-\mu}\wedge P^\mu\,. \tag{3.113}$$

This r-matrix is called the light-cone κ-Poincaré. Again, by introducing the coordinates,

$$x^\pm \equiv \frac{x^0 \pm x^1}{\sqrt{2}}\,, \qquad (\mathrm{d}x^2)^2+(\mathrm{d}x^3)^2 = \mathrm{d}\rho^2+\rho^2\,\mathrm{d}\theta^2\,, \tag{3.114}$$

the YB-deformed background is given by (see section 4.5 of [54])

$$
\begin{aligned}
&\mathrm{d}s^2=\frac{\frac{z^4}{z^4-(\eta x^+)^2}\,(\mathrm{d}z^2-2\mathrm{d}x^+\mathrm{d}x^-)-\eta^2\,\frac{(x^+\mathrm{d}z)^2+(\rho\mathrm{d}x^+)^2-2x^+\rho\mathrm{d}x^+\mathrm{d}\rho}{z^4-(\eta x^+)^2}+\mathrm{d}\rho^2+\rho^2\mathrm{d}\theta^2}{z^2}+\mathrm{d}s^2_{\mathrm{S}^5}\,,\\
&B_2=\frac{\eta\mathrm{d}x^+\wedge(x^+\mathrm{d}x^- -\rho\mathrm{d}\rho)}{z^4-(\eta x^+)^2}\,,\qquad \Phi=\frac{1}{2}\ln\left[\frac{z^4}{z^4-(\eta x^+)^2}\right],\qquad I^-=3\eta,\\
&\hat{\mathsf{F}}_1=0,\quad \hat{\mathsf{F}}_3=-\frac{4\eta\rho}{z^5}\mathrm{d}z\wedge\bigl(\rho\mathrm{d}x^+-x^+\mathrm{d}\rho\bigr)\wedge\mathrm{d}\theta,\quad \hat{\mathsf{F}}_5=4\left[\frac{z^4}{z^4-(\eta x^+)^2}\omega_{\mathrm{AdS}_5}+\omega_{\mathrm{S}^5}\right],\\
&\hat{\mathsf{F}}_7=-\frac{4\eta}{z^4-(\eta x^+)^2}\mathrm{d}x^+\wedge\bigl(x^+\mathrm{d}x^- -\rho\mathrm{d}\rho\bigr)\wedge\omega_{\mathrm{S}^5}\,,\qquad \hat{\mathsf{F}}_9=0.
\end{aligned}
\tag{3.115}
$$

The R-R potentials can be found as follows:

$$
\begin{aligned}
&\hat{\mathsf{C}}_0=0\,, && \hat{\mathsf{C}}_2=\frac{\eta\,\rho}{z^4}\,\bigl(\rho\,\mathrm{d}x^+-x^+\,\mathrm{d}\rho\bigr)\wedge\mathrm{d}\theta\,,\\
&\hat{\mathsf{C}}_4=\frac{\rho}{z^4-(\eta\,x^+)^2}\,\mathrm{d}x^+\wedge\mathrm{d}x^-\wedge\mathrm{d}\rho\wedge\mathrm{d}\theta+\omega_4\,, && \hat{\mathsf{C}}_6=-B_2\wedge\omega_4\,,\quad \hat{\mathsf{C}}_8=0\,.
\end{aligned}
\tag{3.116}
$$

The corresponding dual fields are given by

$$
\begin{aligned}
&\mathrm{d}s^2_{\text{dual}}=\frac{\mathrm{d}z^2-2\mathrm{d}x^+\,\mathrm{d}x^-+\mathrm{d}\rho^2+\rho^2\,\mathrm{d}\theta^2}{z^2}+\mathrm{d}s^2_{\mathrm{S}^5}\,,\qquad \tilde{\phi}=0\,,\\
&\beta=\eta\,\hat{M}_{-\mu}\wedge\hat{P}^\mu=\eta\,\partial_-\wedge(x^+\,\partial_++\rho\,\partial_\rho)=\eta\,\partial_-\wedge(x^+\,\partial_++x^2\,\partial_2+x^3\,\partial_3)\,,
\end{aligned}
\tag{3.117}
$$

and it is easy to check that the divergence formula is satisfied:

$$
I^-=3\,\eta=\tilde{D}_m\beta^{-m}\,. \tag{3.118}
$$

We can calculate other types of the R-R field fields as

$$
\begin{aligned}
&\mathsf{F}_1=0\,, && \mathsf{F}_3=-\frac{4\,\eta\,\rho}{z^5}\,\mathrm{d}z\wedge(\rho\,\mathrm{d}x^+-x^+\,\mathrm{d}\rho)\wedge\mathrm{d}\theta\,,\\
&\mathsf{F}_5=4\,\bigl(\omega_{\mathrm{AdS}_5}+\omega_{\mathrm{S}^5}\bigr)\,, && \mathsf{F}_7=0\,,\qquad \mathsf{F}_9=0\,,\\
&\mathsf{A}_0=0\,, && \mathsf{A}_2=\frac{\eta\,\rho}{z^4}\,\bigl(\rho\,\mathrm{d}x^+-x^+\,\mathrm{d}\rho\bigr)\wedge\mathrm{d}\theta\,,\\
&\mathsf{A}_4=\frac{\rho}{z^4}\,\mathrm{d}x^+\wedge\mathrm{d}x^-\wedge\mathrm{d}\rho\wedge\mathrm{d}\theta+\omega_4\,, && \mathsf{A}_6=0\,,\qquad \mathsf{A}_8=0\,,
\end{aligned}
\tag{3.119}
$$

and

$$
\begin{aligned}
&\check{\mathsf{F}}_1=0\,,\quad \check{\mathsf{F}}_3=0\,,\quad \check{\mathsf{F}}_5=4\,\bigl(\omega_{\mathrm{AdS}_5}+\omega_{\mathrm{S}^5}\bigr)\,, && \check{\mathsf{F}}_7=0\,,\quad \check{\mathsf{F}}_9=0\,,\\
&\check{\mathsf{C}}_0=0\,,\quad \check{\mathsf{C}}_2=0\,,\quad \check{\mathsf{C}}_4=\frac{\rho}{z^4}\,\mathrm{d}x^+\wedge\mathrm{d}x^-\wedge\mathrm{d}\rho\wedge\mathrm{d}\theta+\omega_4\,, && \check{\mathsf{C}}_6=0\,,\quad \check{\mathsf{C}}_8=0\,,
\end{aligned}
\tag{3.120}
$$

and the β-twisted fields are again invariant under the YB deformation.

The non-geometric Q-flux has the non-vanishing components,

$$
Q_+{}^{-+}=Q_2{}^{-2}=Q_3{}^{-3}=\eta\,, \tag{3.121}
$$

and again by compactifying one of the x^1, x^2, and x^3 directions, this background becomes a T-fold. Namely, if we compactify the x^3-direction as, $x^3\sim x^3+\eta^{-1}$, the associated monodromy becomes

$$
\begin{aligned}
&\mathcal{H}_{MN}(x^3+\eta^{-1})=\bigl[\Omega^{\mathrm{T}}\mathcal{H}(x^3)\,\Omega\bigr]_{MN}\,,\quad \Omega^M{}_N\equiv\begin{pmatrix}\delta^m_n & 2\,\delta^{[m}_-\,\delta^{n]}_3\\ 0 & \delta^n_m\end{pmatrix}\,,\\
&\mathsf{F}(x^3+\eta^{-1})=\mathrm{e}^{-\omega\vee}\,\mathsf{F}(x^3)\,,\qquad \omega^{mn}=2\,\delta^{[m}_-\,\delta^{n]}_3\,.
\end{aligned}
\tag{3.122}
$$

4 Conclusion and discussion

In this paper, we have first reviewed the notion of T-folds by showing two examples: (1) a toy model which shows how to obtain a T-fold background upon a chain of dualizations of a geometric torus and (2) the (co-dimension 1) exotic 5^2_2-brane background. These T-folds require the full set of T-duality transformations as transition functions to be globally well-defined.

Then, we have elucidated that the simple formula (3.19) proposed in [33] and the divergence formula (3.11) reproduce various YB-deformed backgrounds. This means that the YB deformation with a classical r-matrix $r = \frac{1}{2}\, r^{ij}\, T_i \wedge T_j$ satisfying the (homogeneous) CYBE, is equivalent to the β-deformation with the deformation parameter

$$r^{mn} = 2\,\eta\, r^{ij}\, \hat{T}_i^m\, \hat{T}_j^n\,. \tag{4.1}$$

We also considered a known background obtained by the non-Abelian T-duality and showed that the extra vector I determined by the divergence formula (3.11) makes the background a solution of GSE.

We have then computed monodromy matrices for various YB-deformed backgrounds and a non-Abelian T-dual background. In order to clarify the general pattern, let us consider a YB deformation associated with a classical r-matrix,

$$r = \frac{1}{2}\left[a^{\mu\nu\rho}\, M_{\mu\nu} \wedge P_\rho + b^\mu\, D \wedge P_\mu\right] \qquad (a^{\mu\nu\rho} = a^{[\mu\nu]\rho},\ b^\mu:\ \text{constant})\,, \tag{4.2}$$

where $b^\mu = 0$ for YB-deformations of Minkowski spacetime, and $a^{\mu\nu\rho}$ and b^μ should be chosen such that r satisfies the homogeneous CYBE. In this case, the β-field in the YB-deformed background becomes

$$\beta = 2\,\eta\, a^{\mu\nu\rho}\, x_\mu\, \partial_\nu \wedge \partial_\rho + \eta\, b^\mu\, (z\,\partial_z + x^\nu\,\partial_\nu) \wedge \partial_\mu\,, \tag{4.3}$$

and this provides the constant Q-flux,

$$Q = \eta\left(2\, a_\mu{}^{\nu\rho} + \delta_\mu^{[\nu}\, b^{\rho]}\right) \mathrm{d}x^\mu \otimes \partial_\nu \wedge \partial_\rho + \eta\, b^\mu\, \mathrm{d}z \otimes \partial_z \wedge \partial_\mu\,. \tag{4.4}$$

By compactifying some of x^μ directions, the background becomes a T-fold. Importantly, as long as the r-matrix solves the homogeneous CYBE, the deformed background is a solution of DFT. Therefore, the YB deformation is a very systematic procedure to obtain solutions with Q-fluxes in DFT. Although we have considered YB deformations of Minkowski and $AdS_5 \times S^5$ backgrounds, it is applicable to more general cases such as $AdS_3 \times S^3 \times S^3 \times S^1$ solutions.

On the other hand, let us remember that the GSE exhibits one isometry direction. This may suggest that they are effectively a 9-dimensional theory. In this respect, as it was denoted in [31], it is still an open problem what is the explicit relation, if any, between the GSE and the 9-dimensional gauged supergravities that involve the gauging of the trombone symmetry of type IIB supergravity [95, 96].[12] If this were the case, then

[12]As it occurs with GSE, gauged supergravities that are obtained by gauging the trombone symmetry or dimensional reduction on non-unimodular group manifolds cannot be derived from an action principle.

JHEP12(2017)108

an additional question is in order. As the trombone symmetry is considered an accidental symmetry which is broken at higher order α'-corrections [95], it would be interesting to seek for the relation between type IIB supergravity and GSE, including α'-corrections.

It is also interesting to study the Poisson-Lie (PL) T-duality in our context. The η-deformation [10, 11, 41, 42], which is an example of YB-deformations, is related to another integrable deformation called the λ-deformation [55, 97–100] via the PL T-duality [101, 102]. This relation has been further generalized by intriguing works [103, 104]. Hence by generalizing our work to include the modified CYBE case, it should be possible to study the PL T-duality in our context. In fact, it is remarkable that the PL T-duality in DFT has been discussed in the recent work [105] from another angle, the global structure of DFT,[13] independently of a series of our works. As a matter of course, these directions meet up at some point.

In summary, we have shed light on a non-geometric aspect of YB deformation. Namely, using the formulas (3.19) and (3.11), we have established a mapping between YB deformations and non-geometric backgrounds involving Q-fluxes. We hope that our result could be the starting point to delve into the relation between integrable deformations and non-geometric backgrounds.

Acknowledgments

We are very grateful to Ursula Carow-Watamura, Yukio Kaneko, Domenico Orlando, Jeong-Hyuck Park, Shigeki Sugimoto, and Satoshi Watamura for valuable discussions. We appreciate useful discussions during the workshops "Geometry, Duality and Strings" at Yukawa Institute for Theoretical Physics and "Noncommutative Geometry, Duality and Quantum Gravity" at Department of Physics, Kyoto University. J.J.F-M gratefully acknowledges the support of JSPS (Postdoctoral Fellowship) and Fundación Séneca/Universidad de Murcia (Programa Saavedra Fajardo). The work of J.S. was supported by the Japan Society for the Promotion of Science (JSPS). The work of K.Y. was supported by the Supporting Program for Interaction-based Initiative Team Studies (SPIRITS) from Kyoto University and by a JSPS Grant-in-Aid for Scientific Research (C) No. 15K05051. This work was also supported in part by the JSPS Japan-Russia Research Cooperative Program.

A Generating GSE solutions with Penrose limits

In this appendix, we consider Penrose limit [68, 69] of YB-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ backgrounds and reproduce solutions of GSE studied in section 3.4. The R-R fluxes in the YB-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ backgrounds may disappear under the Penrose limit. In that case, the resulting backgrounds become purely NS-NS solutions of GSE.

Penrose limit [68, 69] is formulated for the standard supergravity. But, at least so far, there is no general argument on Penrose limit for the GSE case. Hence, it is quite non-trivial whether it can be extended to GSE or not. Here, we will not discuss a general theory

[13] In relation to the global structure, the topology of DFT is discussed in [106].

of Penrose limit for GSE, but explain how to take a scaling of the extra vector I. The point here is that a YB-deformed background contains a deformations parameter and I is proportional to it. Hence, there is a freedom to scale the deformation parameter in taking a Penrose limit. Without scaling the deformation parameter, 5D Minkowski spacetime is obtained as in the undeformed case. On the other hand, by taking an appropriate scaling of the deformation parameter, one can obtain a non-trivial solution of GSE with non-vanishing extra vector fields. We refer to the latter manner as *the modified Penrose limit*. As a result, this modified Penrose limit may be regarded as a technique to generate solutions of GSE.[14]

A.1 Penrose limit of Poincaré AdS_5

Let us first recall how to take a Penrose limit of the Poincaré metric of AdS_5.

The metric is given by

$$\mathrm{d}s^2 = \frac{r^2}{R^2}\left(-\mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) + R^2\,\frac{\mathrm{d}r^2}{r^2}\,, \tag{A.1}$$

where $\vec{x} = (x^1, x^2, x^3)$.

The first task is to determine a null geodesic. Here we are interested in a radial null geodesic described by

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^2 = R^2\,\frac{\dot{r}^2}{r^2} + \frac{r^2}{R^2}\,(-\dot{t}^2) = 0\,. \tag{A.2}$$

Here τ is an affine parameter and the symbol "$\cdot$" denotes a derivative in terms of τ. From the energy conservation, we obtain that

$$\frac{r^2}{R^2}\,\dot{t} \equiv E \quad (\text{constant})\,. \tag{A.3}$$

Hereafter, we will set $E = 1$ by rescaling τ. Then the equation (A.2) can be rewritten as

$$\dot{r}^2 = 1\,. \tag{A.4}$$

Hence, we will take a solution as

$$r = -\tau\,, \tag{A.5}$$

by adjusting an integration constant to be zero. Then t can also be determined as follows:

$$t = -\frac{R^2}{\tau}\,. \tag{A.6}$$

As a result, the radial null geodesic is described as

$$t = \frac{R^2}{r}\,. \tag{A.7}$$

[14] Without any general argument, it is not ensured that the resulting background should satisfy the GSE. However, this point can be overcome by directly checking the GSE for the resulting background. As far as we have checked, it seems likely that this procedure works well.

Let us take a Penrose limit by employing the radial null geodesic (A.7). The first step is to introduce a new variable $\tilde{t}$ as a fluctuation around the null geodesic as

$$t = \frac{R^2}{r} - \tilde{t}\,. \tag{A.8}$$

Then, the metric of Poincaré AdS_5 is rewritten into the pp-wave form:

$$\mathrm{d}s^2 = -2\,\mathrm{d}r\,\mathrm{d}\tilde{t} - \frac{r^2}{R^2}\,\mathrm{d}\tilde{t}^2 + \frac{r^2}{R^2}\,\mathrm{d}\vec{x}^2\,. \tag{A.9}$$

Next, by further transforming the coordinates as

$$\vec{x} = \frac{R}{r}\,\vec{y}\,, \qquad \tilde{t} = v - \frac{1}{2r}\,\vec{y}^2\,, \tag{A.10}$$

the metric can be rewritten as

$$\mathrm{d}s^2 = -2\,\mathrm{d}r\,\mathrm{d}v + \mathrm{d}\vec{y}^2 + \mathcal{O}\big(1/R^2\big)\,. \tag{A.11}$$

Finally, by taking the $R \to \infty$ limit, the metric of 5D Minkowski spacetime is obtained.

A.2 Penrose limits of YB-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$

Our aim here is to consider the modified Penrose limit of YB-deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ with classical r-matrices satisfying the homogeneous CYBE. In the following, we will focus upon two examples of non-unimodular classical r-matrices.

Example 1) [solution of section 3.6.2] $\xrightarrow{\text{Penrose limit}}$ [solution of section 3.4.2]. The first example is a YB-deformed background associated with $r = \frac{1}{2}\,P_0 \wedge D$, which was studied in section 3.6.2. To take a Penrose limit of the background (3.86), let us rescale the fields as follows:

$$\begin{aligned} \mathrm{d}s^2 &\to \quad \mathrm{d}\tilde{s}^2 = R^2\,\mathrm{d}s^2\,, \quad B_2 \to \quad \tilde{B}_2 = R^2\,B_2\,, \\ F_3 &\to \quad \tilde{F}_3 = R^2\,F_3\,, \quad F_5 \to \quad \tilde{F}_5 = R^4\,F_5\,. \end{aligned} \tag{A.12}$$

After performing a coordinate transformation for the radial direction,

$$z = \frac{R^2}{r}\,, \tag{A.13}$$

the radial null geodesic is given by

$$x^0 = \frac{R^2}{r}\,. \tag{A.14}$$

This expression coincides with the one (A.7) even after performing the deformation.

As in the case of Poincaré AdS_5 , a new variable $\tilde{t}$ is introduced as a fluctuation around the null geodesic (A.14):

$$x^0 = \frac{R^2}{r} - \tilde{t}\,. \tag{A.15}$$

Let us perform a further coordinate transformation,

$$\rho = \frac{R}{r}\,p\,, \qquad \tilde{t} = v - \frac{p^2}{2r}\,. \tag{A.16}$$

If the $R \to \infty$ limit is taken naively, one can perform the usual Penrose limit, but it again leads to 5D Minkowski spacetime as in the case of the Poincaré AdS$_5$.

It is interesting to add a modification to the usual process. That is to rescale the deformation parameter η as well,

$$\eta = R^2 \, \xi \, . \tag{A.17}$$

We refer to this modification as the modified Penrose limit.

By taking the $R \to \infty$ limit and also the flat limit of the S^5 part, we obtain the YB-deformed Minkowski background (3.52) with the following identifications:

$$\{x^+, \, x^-, \, x, \, y, \, z, \, \eta\} \quad \longleftrightarrow \quad \{r, \, v, \, \rho \sin\theta \cos\phi, \, \rho \sin\theta \sin\phi, \, z, \, \xi\} \, . \tag{A.18}$$

Remarkably, all of the R-R fluxes have vanished under this modified Penrose limit.

Example 2) [solution of section 3.6.3] $\xrightarrow{\text{Penrose limit}}$ [solution of section 3.4.3]. Let us next consider another YB-deformed background studied in section 3.6.3. To consider a Penrose limit of the background (3.97), let us rescale the fields as follows:

$$\begin{aligned} ds^2 \to \quad d\tilde{s} = R^2 \, ds^2 \, , \quad B_2 \to \quad \tilde{B}_2 = R^2 \, B_2 \, , \\ F_3 \to \quad \tilde{F}_3 = R^2 \, F_3 \, , \quad F_5 \to \quad \tilde{F}_5 = R^4 \, F_5 \, . \end{aligned} \tag{A.19}$$

After performing a coordinate transformation,

$$z = \frac{R^2}{r} \, , \tag{A.20}$$

we obtain a radial null geodesic, which again takes the form,

$$x^0 = \frac{R^2}{r} \, . \tag{A.21}$$

Let us next introduce a new variable $\tilde{t}$ as a fluctuation around the null geodesic (A.21):

$$x^0 = \frac{R^2}{r} - \tilde{t} \, . \tag{A.22}$$

Then, we perform a further coordinate transformation

$$x^1 = \frac{R}{r} \, z \, , \qquad \rho = \frac{R}{r} \, p \, , \qquad \tilde{t} = v - \frac{p^2 + z^2}{2r} \, . \tag{A.23}$$

As in the previous case, the deformation parameter is rescaled as

$$\eta = R^2 \, \xi \, . \tag{A.24}$$

After taking the $R \to \infty$ limit, the resulting background is given by (3.64) with the following replacements:

$$\{r, \, v, \, p, \, \theta, \, \xi\} \to \left\{x^+, \, x^-, \, \sqrt{x^2 + y^2} \, , \arctan(y/x) \, , \eta \right\} . \tag{A.25}$$

Note that all of the R-R fluxes have vanished again as in the previous example (3.52).

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, Int. J. Theor. Phys. **38** (1999) 1113 [hep-th/9711200] [INSPIRE].

[2] N. Beisert et al., *Review of AdS/CFT integrability: an overview*, Lett. Math. Phys. **99** (2012) 3 [arXiv:1012.3982] [INSPIRE].

[3] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in* $AdS_5 \times S^5$ *background*, Nucl. Phys. **B 533** (1998) 109 [hep-th/9805028] [INSPIRE].

[4] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the* $AdS_5 \times S^5$ *superstring*, Phys. Rev. **D 69** (2004) 046002 [hep-th/0305116] [INSPIRE].

[5] C. Klimčík, *Yang-Baxter σ-models and dS/AdS T duality*, JHEP **12** (2002) 051 [hep-th/0210095] [INSPIRE].

[6] C. Klimčík, *On integrability of the Yang-Baxter σ-model*, J. Math. Phys. **50** (2009) 043508 [arXiv:0802.3518] [INSPIRE].

[7] C. Klimčík, *Integrability of the bi-Yang-Baxter σ-model*, Lett. Math. Phys. **104** (2014) 1095 [arXiv:1402.2105] [INSPIRE].

[8] F. Delduc, M. Magro and B. Vicedo, *On classical q-deformations of integrable σ-models*, JHEP **11** (2013) 192 [arXiv:1308.3581] [INSPIRE].

[9] T. Matsumoto and K. Yoshida, *Yang-Baxter σ-models based on the CYBE*, Nucl. Phys. **B 893** (2015) 287 [arXiv:1501.03665] [INSPIRE].

[10] F. Delduc, M. Magro and B. Vicedo, *An integrable deformation of the* $AdS_5 \times S^5$ *superstring action*, Phys. Rev. Lett. **112** (2014) 051601 [arXiv:1309.5850] [INSPIRE].

[11] F. Delduc, M. Magro and B. Vicedo, *Derivation of the action and symmetries of the q-deformed* $AdS_5 \times S^5$ *superstring*, JHEP **10** (2014) 132 [arXiv:1406.6286] [INSPIRE].

[12] I. Kawaguchi, T. Matsumoto and K. Yoshida, *Jordanian deformations of the* $AdS_5 \times S^5$ *superstring*, JHEP **04** (2014) 153 [arXiv:1401.4855] [INSPIRE].

[13] B.E. Fridling and A. Jevicki, *Dual representations and ultraviolet divergences in nonlinear σ models*, Phys. Lett. **B 134** (1984) 70 [INSPIRE].

[14] E.S. Fradkin and A.A. Tseytlin, *Quantum equivalence of dual field theories*, Annals Phys. **162** (1985) 31 [INSPIRE].

[15] X.C. de la Ossa and F. Quevedo, *Duality symmetries from non-Abelian isometries in string theory*, Nucl. Phys. **B 403** (1993) 377 [hep-th/9210021] [INSPIRE].

[16] M. Gasperini, R. Ricci and G. Veneziano, *A problem with non-Abelian duality?*, Phys. Lett. **B 319** (1993) 438 [hep-th/9308112] [INSPIRE].

[17] A. Giveon and M. Roček, *On non-Abelian duality*, Nucl. Phys. **B 421** (1994) 173 [hep-th/9308154] [INSPIRE].

[18] E. Alvarez, L. Álvarez-Gaumé and Y. Lozano, *On non-Abelian duality*, Nucl. Phys. **B 424** (1994) 155 [hep-th/9403155] [INSPIRE].

[19] S. Elitzur, A. Giveon, E. Rabinovici, A. Schwimmer and G. Veneziano, *Remarks on non-Abelian duality*, Nucl. Phys. **B 435** (1995) 147 [hep-th/9409011] [INSPIRE].

[20] K. Sfetsos and D.C. Thompson, *On non-Abelian T-dual geometries with Ramond fluxes*, Nucl. Phys. **B 846** (2011) 21 [arXiv:1012.1320] [INSPIRE].

[21] Y. Lozano, E. Ó Colgáin, K. Sfetsos and D.C. Thompson, *Non-Abelian T-duality, Ramond fields and coset geometries*, JHEP **06** (2011) 106 [arXiv:1104.5196] [INSPIRE].

[22] G. Itsios, C. Núñez, K. Sfetsos and D.C. Thompson, *Non-Abelian T-duality and the AdS/CFT correspondence: new $N = 1$ backgrounds*, Nucl. Phys. **B 873** (2013) 1 [arXiv:1301.6755] [INSPIRE].

[23] T. Matsumoto and K. Yoshida, *Integrability of classical strings dual for noncommutative gauge theories*, JHEP **06** (2014) 163 [arXiv:1404.3657] [INSPIRE].

[24] S.J. van Tongeren, *Almost Abelian twists and AdS/CFT*, Phys. Lett. **B 765** (2017) 344 [arXiv:1610.05677] [INSPIRE].

[25] T. Araujo, I. Bakhmatov, E. Ó Colgáin, J. Sakamoto, M.M. Sheikh-Jabbari and K. Yoshida, *Yang-Baxter σ-models, conformal twists and noncommutative Yang-Mills theory*, Phys. Rev. **D 95** (2017) 105006 [arXiv:1702.02861] [INSPIRE].

[26] T. Araujo, I. Bakhmatov, E. Ó Colgáin, J.-I. Sakamoto, M.M. Sheikh-Jabbari and K. Yoshida, *Conformal twists, Yang-Baxter σ-models & holographic noncommutativity*, arXiv:1705.02063 [INSPIRE].

[27] B. Hoare and A.A. Tseytlin, *Homogeneous Yang-Baxter deformations as non-Abelian duals of the AdS_5 σ-model*, J. Phys. **A 49** (2016) 494001 [arXiv:1609.02550] [INSPIRE].

[28] R. Borsato and L. Wulff, *Integrable deformations of T-dual σ models*, Phys. Rev. Lett. **117** (2016) 251602 [arXiv:1609.09834] [INSPIRE].

[29] B. Hoare and D.C. Thompson, *Marginal and non-commutative deformations via non-Abelian T-duality*, JHEP **02** (2017) 059 [arXiv:1611.08020] [INSPIRE].

[30] Y. Sakatani, S. Uehara and K. Yoshida, *Generalized gravity from modified DFT*, JHEP **04** (2017) 123 [arXiv:1611.05856] [INSPIRE].

[31] A. Baguet, M. Magro and H. Samtleben, *Generalized IIB supergravity from exceptional field theory*, JHEP **03** (2017) 100 [arXiv:1612.07210] [INSPIRE].

[32] J.-I. Sakamoto, Y. Sakatani and K. Yoshida, *Weyl invariance for generalized supergravity backgrounds from the doubled formalism*, PTEP **2017** (2017) 053B07 [arXiv:1703.09213] [INSPIRE].

[33] J.-I. Sakamoto, Y. Sakatani and K. Yoshida, *Homogeneous Yang-Baxter deformations as generalized diffeomorphisms*, J. Phys. **A 50** (2017) 415401 [arXiv:1705.07116] [INSPIRE].

[34] W. Siegel, *Two vierbein formalism for string inspired axionic gravity*, Phys. Rev. **D 47** (1993) 5453 [hep-th/9302036] [INSPIRE].

[35] W. Siegel, *Superspace duality in low-energy superstrings*, Phys. Rev. **D 48** (1993) 2826 [hep-th/9305073] [INSPIRE].

[36] W. Siegel, *Manifest duality in low-energy superstrings*, in *International Conference on Strings* 93, Berkeley CA U.S.A., 24–29 May 1993, pg. 353 [hep-th/9308133] [INSPIRE].

[37] C. Hull and B. Zwiebach, *Double field theory*, JHEP **09** (2009) 099 [arXiv:0904.4664] [INSPIRE].

[38] O. Hohm, C. Hull and B. Zwiebach, *Generalized metric formulation of double field theory*, *JHEP* **08** (2010) 008 [arXiv:1006.4823] [INSPIRE].

[39] V.G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, *Sov. Math. Dokl.* **32** (1985) 254 [*Dokl. Akad. Nauk Ser. Fiz.* **283** (1985) 1060] [INSPIRE].

[40] M. Jimbo, *A q difference analog of $U(g)$ and the Yang-Baxter equation*, *Lett. Math. Phys.* **10** (1985) 63 [INSPIRE].

[41] G. Arutyunov, R. Borsato and S. Frolov, *S-matrix for strings on η-deformed $AdS_5 \times S^5$*, *JHEP* **04** (2014) 002 [arXiv:1312.3542] [INSPIRE].

[42] G. Arutyunov, R. Borsato and S. Frolov, *Puzzles of η-deformed $AdS_5 \times S^5$*, *JHEP* **12** (2015) 049 [arXiv:1507.04239] [INSPIRE].

[43] T. Matsumoto and K. Yoshida, *Lunin-Maldacena backgrounds from the classical Yang-Baxter equation — towards the gravity/CYBE correspondence*, *JHEP* **06** (2014) 135 [arXiv:1404.1838] [INSPIRE].

[44] T. Matsumoto and K. Yoshida, *Schrödinger geometries arising from Yang-Baxter deformations*, *JHEP* **04** (2015) 180 [arXiv:1502.00740] [INSPIRE].

[45] I. Kawaguchi, T. Matsumoto and K. Yoshida, *A Jordanian deformation of AdS space in type IIB supergravity*, *JHEP* **06** (2014) 146 [arXiv:1402.6147] [INSPIRE].

[46] T. Matsumoto and K. Yoshida, *Yang-Baxter deformations and string dualities*, *JHEP* **03** (2015) 137 [arXiv:1412.3658] [INSPIRE].

[47] T. Matsumoto and K. Yoshida, *Integrable deformations of the $AdS_5 \times S^5$ superstring and the classical Yang-Baxter equation — towards the gravity/CYBE correspondence*, *J. Phys. Conf. Ser.* **563** (2014) 012020 [arXiv:1410.0575] [INSPIRE].

[48] T. Matsumoto and K. Yoshida, *Towards the gravity/CYBE correspondence — the current status*, *J. Phys. Conf. Ser.* **670** (2016) 012033 [INSPIRE].

[49] S.J. van Tongeren, *On classical Yang-Baxter based deformations of the $AdS_5 \times S^5$ superstring*, *JHEP* **06** (2015) 048 [arXiv:1504.05516] [INSPIRE].

[50] S.J. van Tongeren, *Yang-Baxter deformations, AdS/CFT and twist-noncommutative gauge theory*, *Nucl. Phys.* **B 904** (2016) 148 [arXiv:1506.01023] [INSPIRE].

[51] T. Kameyama, H. Kyono, J.-I. Sakamoto and K. Yoshida, *Lax pairs on Yang-Baxter deformed backgrounds*, *JHEP* **11** (2015) 043 [arXiv:1509.00173] [INSPIRE].

[52] H. Kyono and K. Yoshida, *Supercoset construction of Yang-Baxter deformed $AdS_5 \times S^5$ backgrounds*, *PTEP* **2016** (2016) 083B03 [arXiv:1605.02519] [INSPIRE].

[53] B. Hoare and S.J. van Tongeren, *On Jordanian deformations of AdS_5 and supergravity*, *J. Phys.* **A 49** (2016) 434006 [arXiv:1605.03554] [INSPIRE].

[54] D. Orlando, S. Reffert, J.-I. Sakamoto and K. Yoshida, *Generalized type IIB supergravity equations and non-Abelian classical r-matrices*, *J. Phys.* **A 49** (2016) 445403 [arXiv:1607.00795] [INSPIRE].

[55] R. Borsato and L. Wulff, *Target space supergeometry of η and λ-deformed strings*, *JHEP* **10** (2016) 045 [arXiv:1608.03570] [INSPIRE].

[56] D. Osten and S.J. van Tongeren, *Abelian Yang-Baxter deformations and TsT transformations*, *Nucl. Phys.* **B 915** (2017) 184 [arXiv:1608.08504] [INSPIRE].

[57] O. Lunin and J.M. Maldacena, *Deforming field theories with* $U(1) \times U(1)$ *global symmetry and their gravity duals*, *JHEP* **05** (2005) 033 [hep-th/0502086] [INSPIRE].

[58] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, *JHEP* **05** (2005) 069 [hep-th/0503201] [INSPIRE].

[59] A. Hashimoto and N. Itzhaki, *Noncommutative Yang-Mills and the AdS/CFT correspondence*, *Phys. Lett.* **B 465** (1999) 142 [hep-th/9907166] [INSPIRE].

[60] J.M. Maldacena and J.G. Russo, *Large-N limit of noncommutative gauge theories*, *JHEP* **09** (1999) 025 [hep-th/9908134] [INSPIRE].

[61] C.P. Herzog, M. Rangamani and S.F. Ross, *Heating up Galilean holography*, *JHEP* **11** (2008) 080 [arXiv:0807.1099] [INSPIRE].

[62] J. Maldacena, D. Martelli and Y. Tachikawa, *Comments on string theory backgrounds with non-relativistic conformal symmetry*, *JHEP* **10** (2008) 072 [arXiv:0807.1100] [INSPIRE].

[63] A. Adams, K. Balasubramanian and J. McGreevy, *Hot spacetimes for cold atoms*, *JHEP* **11** (2008) 059 [arXiv:0807.1111] [INSPIRE].

[64] G. Arutyunov, S. Frolov, B. Hoare, R. Roiban and A.A. Tseytlin, *Scale invariance of the* η*-deformed* $AdS_5 \times S^5$ *superstring, T-duality and modified type-II equations*, *Nucl. Phys.* **B 903** (2016) 262 [arXiv:1511.05795] [INSPIRE].

[65] L. Wulff and A.A. Tseytlin, κ*-symmetry of superstring* σ*-model and generalized* 10d *supergravity equations*, *JHEP* **06** (2016) 174 [arXiv:1605.04884] [INSPIRE].

[66] M.J. Duff, *Duality rotations in string theory*, *Nucl. Phys.* **B 335** (1990) 610 [INSPIRE].

[67] C.M. Hull, *A geometry for non-geometric string backgrounds*, *JHEP* **10** (2005) 065 [hep-th/0406102] [INSPIRE].

[68] R. Penrose, *Any space-time has a plane wave as a limit*, in *Differential geometry and relativity: a volume in honour of André Lichnerowicz on his* 60$^{\text{th}}$ *birthday*, Springer, Dordrecht The Netherlands, (1976), pg. 271.

[69] R. Güven, *Plane wave limits and T duality*, *Phys. Lett.* **B 482** (2000) 255 [hep-th/0005061] [INSPIRE].

[70] S. Kachru, M.B. Schulz, P.K. Tripathy and S.P. Trivedi, *New supersymmetric string compactifications*, *JHEP* **03** (2003) 061 [hep-th/0211182] [INSPIRE].

[71] J. Shelton, W. Taylor and B. Wecht, *Nongeometric flux compactifications*, *JHEP* **10** (2005) 085 [hep-th/0508133] [INSPIRE].

[72] F. Hassler and D. Lüst, *Non-commutative/non-associative IIA (IIB) Q- and R-branes and their intersections*, *JHEP* **07** (2013) 048 [arXiv:1303.1413] [INSPIRE].

[73] G. Dibitetto, J.J. Fernandez-Melgarejo, D. Marques and D. Roest, *Duality orbits of non-geometric fluxes*, *Fortsch. Phys.* **60** (2012) 1123 [arXiv:1203.6562] [INSPIRE].

[74] A.D. Shapere and F. Wilczek, *Selfdual models with theta terms*, *Nucl. Phys.* **B 320** (1989) 669 [INSPIRE].

[75] A. Giveon, E. Rabinovici and G. Veneziano, *Duality in string background space*, *Nucl. Phys.* **B 322** (1989) 167 [INSPIRE].

[76] A.A. Tseytlin, *Duality symmetric formulation of string world sheet dynamics*, *Phys. Lett.* **B 242** (1990) 163 [INSPIRE].

JHEP12(2017)108

[77] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, *Phys. Rept.* **244** (1994) 77 [hep-th/9401139] [INSPIRE].

[78] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142] [INSPIRE].

[79] D. Andriot, M. Larfors, D. Lüst and P. Patalong, *A ten-dimensional action for non-geometric fluxes*, *JHEP* **09** (2011) 134 [arXiv:1106.4015] [INSPIRE].

[80] M. Graña, R. Minasian, M. Petrini and D. Waldram, *T-duality, generalized geometry and non-geometric backgrounds*, *JHEP* **04** (2009) 075 [arXiv:0807.4527] [INSPIRE].

[81] E. Lozano-Tellechea and T. Ortín, *7-branes and higher Kaluza-Klein branes*, *Nucl. Phys.* **B 607** (2001) 213 [hep-th/0012051] [INSPIRE].

[82] J. de Boer and M. Shigemori, *Exotic branes and non-geometric backgrounds*, *Phys. Rev. Lett.* **104** (2010) 251603 [arXiv:1004.2521] [INSPIRE].

[83] J. Sakamoto, Y. Sakatani and K. Yoshida, work in preparation.

[84] T. Araujo, E. Ó Colgáin, J. Sakamoto, M.M. Sheikh-Jabbari and K. Yoshida, *I in generalized supergravity*, *Eur. Phys. J.* **C 77** (2017) 739 [arXiv:1708.03163] [INSPIRE].

[85] T. Matsumoto, D. Orlando, S. Reffert, J.-I. Sakamoto and K. Yoshida, *Yang-Baxter deformations of Minkowski spacetime*, *JHEP* **10** (2015) 185 [arXiv:1505.04553] [INSPIRE].

[86] A. Borowiec, H. Kyono, J. Lukierski, J.-I. Sakamoto and K. Yoshida, *Yang-Baxter σ-models and Lax pairs arising from κ-Poincaré r-matrices*, *JHEP* **04** (2016) 079 [arXiv:1510.03083] [INSPIRE].

[87] A.A. Tseytlin, *Melvin solution in string theory*, *Phys. Lett.* **B 346** (1995) 55 [hep-th/9411198] [INSPIRE].

[88] G.W. Gibbons and K.-I. Maeda, *Black holes and membranes in higher dimensional theories with dilaton fields*, *Nucl. Phys.* **B 298** (1988) 741 [INSPIRE].

[89] A. Hashimoto and K. Thomas, *Dualities, twists and gauge theories with non-constant non-commutativity*, *JHEP* **01** (2005) 033 [hep-th/0410123] [INSPIRE].

[90] E. Malek, *U-duality in three and four dimensions*, *Int. J. Mod. Phys.* **A 32** (2017) 1750169 [arXiv:1205.6403] [INSPIRE].

[91] E. Malek, *Timelike U-dualities in generalised geometry*, *JHEP* **11** (2013) 185 [arXiv:1301.0543] [INSPIRE].

[92] O. Hohm and B. Zwiebach, *Large gauge transformations in double field theory*, *JHEP* **02** (2013) 075 [arXiv:1207.4198] [INSPIRE].

[93] K. Lee and J.-H. Park, *Covariant action for a string in "doubled yet gauged" spacetime*, *Nucl. Phys.* **B 880** (2014) 134 [arXiv:1307.8377] [INSPIRE].

[94] K. Morand and J.-H. Park, *Classification of non-Riemannian doubled-yet-gauged spacetime*, *Eur. Phys. J.* **C 77** (2017) 685 [arXiv:1707.03713] [INSPIRE].

[95] E. Bergshoeff, T. de Wit, U. Gran, R. Linares and D. Roest, *(Non)Abelian gauged supergravities in nine-dimensions*, *JHEP* **10** (2002) 061 [hep-th/0209205] [INSPIRE].

[96] J.J. Fernandez-Melgarejo, T. Ortín and E. Torrente-Lujan, *The general gaugings of maximal $D = 9$ supergravity*, *JHEP* **10** (2011) 068 [arXiv:1106.1760] [INSPIRE].

[97] K. Sfetsos, *Integrable interpolations: from exact CFTs to non-Abelian T-duals*, Nucl. Phys. **B 880** (2014) 225 [arXiv:1312.4560] [INSPIRE].

[98] S. Demulder, K. Sfetsos and D.C. Thompson, *Integrable λ-deformations: squashing coset CFTs and $AdS_5 \times S^5$*, JHEP **07** (2015) 019 [arXiv:1504.02781] [INSPIRE].

[99] T.J. Hollowood, J.L. Miramontes and D.M. Schmidtt, *An integrable deformation of the $AdS_5 \times S^5$ superstring*, J. Phys. **A 47** (2014) 495402 [arXiv:1409.1538] [INSPIRE].

[100] T.J. Hollowood, J.L. Miramontes and D.M. Schmidtt, *Integrable deformations of strings on symmetric spaces*, JHEP **11** (2014) 009 [arXiv:1407.2840] [INSPIRE].

[101] B. Vicedo, *Deformed integrable σ-models, classical R-matrices and classical exchange algebra on Drinfel'd doubles*, J. Phys. **A 48** (2015) 355203 [arXiv:1504.06303] [INSPIRE].

[102] B. Hoare and A.A. Tseytlin, *On integrable deformations of superstring σ-models related to $AdS_n \times S^n$ supercosets*, Nucl. Phys. **B 897** (2015) 448 [arXiv:1504.07213] [INSPIRE].

[103] K. Sfetsos, K. Siampos and D.C. Thompson, *Generalised integrable λ- and η-deformations and their relation*, Nucl. Phys. **B 899** (2015) 489 [arXiv:1506.05784] [INSPIRE].

[104] Y. Chervonyi and O. Lunin, *Generalized λ-deformations of $AdS_p \times S^p$*, Nucl. Phys. **B 913** (2016) 912 [arXiv:1608.06641] [INSPIRE].

[105] F. Hassler, *Poisson-Lie T-duality in double field theory*, arXiv:1707.08624 [INSPIRE].

[106] F. Hassler, *The topology of double field theory*, arXiv:1611.07978 [INSPIRE].

JHEP12(2017)108