Random Geometric Graph Diameter in the Unit Disk with ℓ_p Metric

Extended Abstract

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Abstract. Let n be a positive integer, $\lambda > 0$ a real number, and $1 \le p \le \infty$. We study the *unit disk random geometric graph* $G_p(\lambda, n)$, defined to be the random graph on n vertices, independently distributed uniformly in the standard unit disk in $\mathbb{R}^2$, with two vertices adjacent if and only if their ℓ_p-distance is at most λ. Let $\lambda = c\sqrt{\ln n/n}$, and let a_p be the ratio of the (Lebesgue) areas of the ℓ_p- and ℓ_2-unit disks. Almost always, $G_p(\lambda, n)$ has no isolated vertices and is also connected if $c > a_p^{-1/2}$, and has $n^{1-a_p c^2}(1 + o(1))$ isolated vertices if $c < a_p^{-1/2}$. Furthermore, we find upper bounds (involving λ but independent of p) for the diameter of $G_p(\lambda, n)$, building on a method originally due to M. Penrose.

1 Introduction

Let D be the Euclidean unit disk in $\mathbb{R}^2$ and n a positive integer. Let V_n be a set of n points in D, distributed independently and uniformly with respect to the usual Lebesgue measure on $\mathbb{R}^2$. For $p \in [1, \infty]$, the ℓ_p *metric* on $\mathbb{R}^2$ is defined by

$$d_p((x_1, y_1), (x_2, y_2)) = \begin{cases} (|x_2 - x_1|^p + |y_2 - y_1|^p)^{1/p} & \text{when } p \in [1, \infty) \ , \\ \max\{|x_2 - x_1|, |y_2 - y_1|\} & \text{when } p = \infty \ . \end{cases}$$

For $\lambda \in (0, \infty)$, the *unit disk random geometric graph* $G_p(\lambda, n)$ on the vertex set V_n is defined by declaring two vertices $u, v \in V_n$ to be adjacent if and only if $d_p(u, v) \le \lambda$. In addition to their theoretical interest, random geometric graphs have important applications to wireless communication networks; see, e.g., [1–3].

Together with X. Jia, the first and third authors studied the case $p = 2$ in [4]. In this extended abstract, we generalize to arbitrary p those results of [4]

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concerning connectedness and graph diameter. Complete results with proofs will be included in a forthcoming paper.

We will say that $G_p(\lambda, n)$ has a property P *almost always* if

$$\lim_{n\to\infty} \Pr\left[G_p(\lambda, n) \text{ has the property } P\right] = 1 \ .$$

Denote by $B_p(u, r)$ the ℓ_p-ball of radius r with center $u \in \mathbb{R}^2$. It is not hard to show that the area of $B_p(u, r)$ is $4r^2\Gamma((p+1)/p)^2/\Gamma((p+2)/p)$, where $\Gamma(\cdot)$ is the usual gamma function. We omit the calculation, which uses the beta function; see [5, §12.4]. An important quantity in our work will be the ratio

$$a_p := \frac{\mathrm{Area}(B_p(u,r))}{\mathrm{Area}(B_2(u,r))} = \frac{4\Gamma\left(\frac{p+1}{p}\right)^2}{\pi\Gamma\left(\frac{p+2}{p}\right)} \ .$$

By another elementary calculation, the ℓ_p-diameter of the unit disk D is

$$\mathrm{diam}_p(D) := \max_{u,v\in D}\{d_p(u,v)\} = \begin{cases} 2^{1/2+1/p} & \text{when } 1 \le p \le 2 \ , \\ 2 & \text{when } 2 \le p \le \infty \ . \end{cases}$$

The diameter is achieved by the points $(\sqrt{2}/2, \sqrt{2}/2)$ and $(-\sqrt{2}/2, -\sqrt{2}/2)$ when $1 \le p \le 2$, and by $(0, 1)$ and $(0, -1)$ when $2 \le p \le \infty$.

Let $\lambda = c\sqrt{\ln n/n}$. In Sect. 2, we show that almost always, $G_p(\lambda, n)$ has $n^{1-a_pc^2}(1+o(1))$ isolated vertices when $c < a_p^{-1/2}$ and no isolated vertices when $c > a_p^{-1/2}$. Penrose [6] has shown that, almost always, $G_p(\lambda, n)$ is connected when it has no isolated points; combining this with our result, it follows that when $c > a_p^{-1/2}$, the graph $G_p(\lambda, n)$ is almost always connected.

The *diameter* of a graph G, denoted $\mathrm{diam}(G)$, is defined as the maximum distance in G between any two of its vertices. This graph-theoretic quantity should not be confused with the diameter of a geometric object with respect to the ℓ_p-metric; we will always denote the latter by diam_p. In Sect. 3, we show that if $c > a_p^{-1/2}$, then almost always $\mathrm{diam}(G_p(\lambda, n)) \le K/\lambda$, where $K \approx 387.17\ldots$ is a constant independent of p. In Sect. 4, we show that when c is larger than a constant depending only on p, we have almost always $\mathrm{diam}(G_p(\lambda, n)) \le 2 \cdot \mathrm{diam}_p(D)(1 + o(1))/\lambda$. In fact, there is a function $c_p(\delta) > 0$ with the following property: if $c > c_p(\delta)$, then almost always $\mathrm{diam}(G_p(\lambda, n)) \le \mathrm{diam}_p(D)(1 + \delta + o(1))/\lambda$.

2 Isolated Vertices

Theorem 1. *Let* $1 \le p \le \infty$, *let* $\lambda = c\sqrt{\ln n/n}$, *and let* X *be the number of isolated vertices in* $G_p(\lambda, n)$. *Then, almost always,*

$$X = \begin{cases} 0 & \textit{when } c > a_p^{-1/2}, \\ n^{1-a_pc^2}(1+o(1)) & \textit{when } 0 < c < a_p^{-1/2} \ . \end{cases}$$

We sketch the proof, which uses the *second moment method* [7] to show that the expected number of isolated vertices is $\mathbb{E}[X] = n^{1-a_p c^2}$, and that the variance is $\mathrm{Var}[X] = o(\mathbb{E}[X]^2)$. When $c < a_p^{-1/2}$, an application of Chebyshev's inequality yields $X = n^{1-a_p c^2}(1 + o(1))$. Let A_i be the event that vertex v_i has degree 0. Then

$$\frac{a_p}{2}\pi\lambda^2(1 + O(\lambda)) \leq \mathrm{Area}\,(B_p(v_i, \lambda) \cap D) \leq a_p\pi\lambda^2 \ ,$$

where the upper (resp. lower) bound is achieved when $B_p(v_i, \lambda) \subseteq D$ (resp. $B_p(v_i, \lambda) \not\subseteq D$). Conditioning on the event that $B_p(v_i, \lambda) \subseteq D$, we have

$$\begin{aligned}(1 - a_p\lambda^2)^{n-1} \leq \Pr[A_i] \leq \Pr[B_p(v_i, \lambda) \subseteq D](1 - a_p\lambda^2)^{n-1}\\ + \Pr[B_p(v_i, \lambda) \not\subseteq D]\left(1 - \frac{a_p}{2}\lambda^2(1 + O(\lambda))\right)^{n-1} \ .\end{aligned}$$

By linearity of expectation, $\mathbb{E}[X] = n \cdot \Pr[A_i] = n^{1-a_p c^2}(1 + o(1))$. The variance is $\mathrm{Var}[X] = O\left(n^{\frac{3}{2}-\frac{3}{2}a_p c^2}\sqrt{\ln n}\right)$, computed via $\Pr[A_i \wedge A_j]$, conditioned on $d_p(v_i, v_j)$. The rest of the proof is a straightforward computation.

Penrose [6, Thm. 1.1] showed that for every $t \geq 0$, the d-dimensional unit-*cube* random geometric graph simultaneously becomes $(t + 1)$-connected and achieves minimum degree $t + 1$. Penrose's proof remains valid for the unit disk. The precise statement is as follows: for $t \geq 0$ and $1 < p \leq \infty$, almost always,

$$\begin{aligned}\min\{\lambda \mid G_p(\lambda, n) \text{ is } (t+1)\text{-connected}\}\\ = \min\{\lambda \mid G_p(\lambda, n) \text{ has minimum degree } t + 1\} \ .\end{aligned}$$

Penrose's proof also works for $p = 1$ in dimension 2, though not for arbitrary dimension d. Combining Penrose's theorem for $t = 0$ with Theorem 1 yields the following.

Theorem 2. *Let* $1 \leq p \leq \infty$ *and* $\lambda = c\sqrt{\ln n/n}$. *Suppose that* $c > a_p^{-1/2}$. *Then, almost always, the unit disk random geometric graph* $G_p(\lambda, n)$ *is connected.*

3 Diameter of $G_p(\lambda, n)$ near the Connectivity Threshold

Suppose that $G_p(\lambda, n)$ is connected by virtue of Theorem 2. Usually, $G_p(\lambda, n)$ will contain two vertices whose ℓ_p-distance is close to $\mathrm{diam}_p(D)$, so that the graph has diameter at least $\mathrm{diam}_p(D)/\lambda$. It appears to be much more difficult to obtain an *upper* bound on diameter. However, there is an upper bound which is a constant multiple of the lower bound, as we now explain.

Theorem 3. *Let* $1 \leq p \leq \infty$ *and* $\lambda = c\sqrt{\ln n/n}$, *where* $c > a_p^{-1/2}$. *Suppose that* $K > 256\sqrt{2} + 8\pi \approx 387.17\ldots$. *Then, almost always,* $\mathrm{diam}(G_p(\lambda, n)) < K/\lambda$.

We sketch the proof of this theorem. For any two points $u, v \in D$, define

$$T_{u,v}(k) := \left(\text{convex hull of } B_2(u, k\lambda) \cup B_2(v, k\lambda)\right) \cap D \ .$$

We impose upon this lozenge-shaped region a grid composed of squares with side length proportional to λ. Let $A_n(k)$ be the event that there exist two points $u, v \in V_n$ such that

(i) at least one of u, v lies in $B_2(O, 1-(k+\sqrt{2})\lambda)$, and
(ii) there is no path in $G_p(\lambda, n)$ joining u to v that lies entirely inside $T_{u,v}(k)$.

We claim that

$$\text{if } k > 128/(\pi\sqrt{2}) \approx 28.180\ldots, \text{ then } \lim_{n\to\infty} \Pr[A_n(k)] = 0. \tag{1}$$

Indeed, if the event $A_n(k)$ occurs, then by Penrose's argument [6, p. 162], there exists a curve L separating u and v which intersects a large number of grid squares, none of which contains any vertex of V_n (see Fig. 1). Combining this fact with a Peierls argument, as in [8, Lemma 3], leads to the bound on k given in (1).

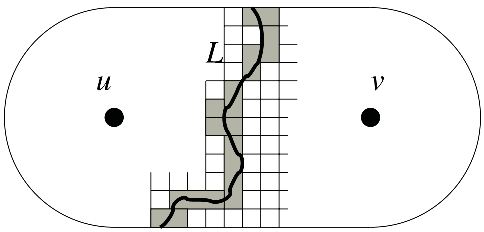

Fig. 1. Two vertices $u, v \in V_n$ which are not connected by any path in $T_{u,v}(k)$, and the "frontier" L separating them.

Let $u, v \in V_n$. If k is large enough, then (1) guarantees the existence of a path from u to v inside $T_{u,v}(k)$. Comparing the total area of $T_{u,v}(k)$ to the area of the ℓ_p-balls around the vertices in a shortest path from u to v inside $T_{u,v}(k)$, one obtains the desired diameter bound on $G_p(\lambda, n)$, completing the proof. (Minor adjustments are needed if u or v is close to the boundary of D.)

Corollary 1. *Let $1 \le p \le \infty$ and $\lambda = c\sqrt{\ln n/n}$, where $c > a_p^{-1/2}$. Suppose that $K > 256\sqrt{2} + 8\pi \approx 387.17\ldots$. Then, almost always, every two vertices u, v in the unit disk random geometric graph $G_p(\lambda, n)$ are joined by a path of length at most $Kd_p(u, v)/\lambda$ in $G_p(\lambda, n)$.*

4 Diameter of $G_p(\lambda, n)$ for Larger c

By means of a "spoke overlay" construction, we improve the upper bound in Theorem 3 by increasing the constant c slightly and reducing the constant K substantially. Roughly, a spoke consists of a number of evenly spaced, overlapping ℓ_p-balls whose centers lie on a diameter L of the Euclidean unit disk D. We superimpose several spokes on D so that the regions of intersection of the ℓ_p-balls are distributed fairly evenly around D. The idea is that if the constant

c is large enough, then, almost always, every region of intersection contains at least one vertex of V_n, so that $G_p(\lambda, n)$ contains a path joining vertices near the antipodes of D on L. The lengths of such paths, which may be calculated geometrically, give an upper bound for the diameter of $G_p(\lambda, n)$.

Definition 1 (Spoke construction). *Fix* $1 \leq p \leq \infty$, $\theta \in (-\pi/2, \pi/2]$, *and* $r > 0$. *Let* D *be the Euclidean unit disk. For* $m \in \mathbb{Z}$, *put*

$$u_m = u_m(r, \theta) = ((r/2 + rm)\cos\theta,\ (r/2 + rm)\sin\theta) \in \mathbb{R}^2 \ .$$

The corresponding spoke *is defined to be the point set* $U_{p,\theta}(r) = \{u_m\} \cap D$, *together with a collection of* ℓ_p*-balls of radius* $\lambda/2$, *one centered at each point* $u_m \in U_{p,\theta}(r)$.

The points u_m lie on the line L_θ through O at angle θ, and the Euclidean distance $d_2(u_m, u_{m'})$ equals $r|m - m'|$. By choosing r sufficiently small, we can ensure that each pair of adjacent ℓ_p-balls intersects in a set with positive area (the shaded rectangles in Fig. 2). Thus the two outermost points on each spoke are joined by a segmented path of Euclidean length approximately 2, which has approximately $2 \cdot \mathrm{diam}_p(D)/\lambda$ edges when $r = \min\{\lambda 2^{-1/2-1/p}, \lambda/2\}$.

Define $A_p^*(r, \lambda/2)$ to be the minimum area of intersection between two ℓ_p-balls in $\mathbb{R}^2$ of radius $\lambda/2$ whose centers are at Euclidean distance r. The general formula for this quantity seems to involve integrals that cannot be evaluated exactly, except for very special cases such as $p = 1, 2, \infty$. However, for fixed r, it is certainly true that $A_p^*(r, \lambda/2) = \Theta(\lambda^2)$.

Theorem 4. *Let* $1 \leq p \leq \infty$, $\lambda = c\sqrt{\ln n/n}$, *and* $r = \min\{\lambda 2^{-1/2-1/p}, \lambda/2\}$. *Suppose that*

$$c > \sqrt{\pi\lambda^2/(2A_p^*(r, \lambda/2))} \ . \qquad (2)$$

Then, almost always, as $n \to \infty$,

$$\mathrm{diam}(G_p(\lambda, n)) \leq (2 \cdot \mathrm{diam}_p(D) + o(1))/\lambda.$$

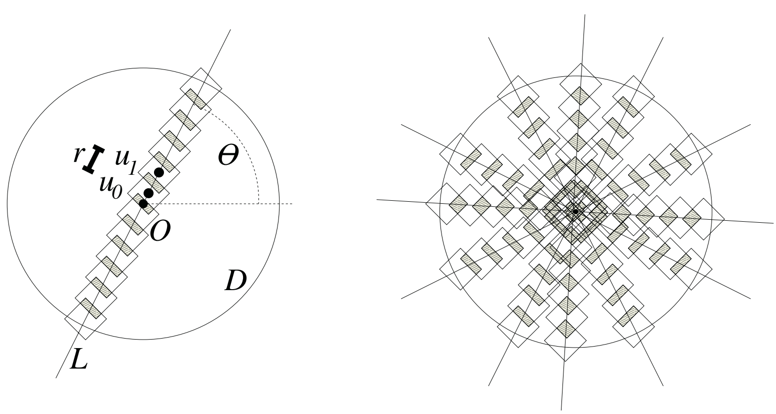

Fig. 2. The spoke overlay construction with $p = 1$, in the unit disk D. The left-hand figure shows a single spoke with parameters r, L, θ. The right-hand figure shows how spokes at different angles are superimposed on D.

Since $A_p^*(r, \lambda/2) = \Theta(\lambda^2)$, the lower bound (2) for c depends only on p.

We sketch the proof of Theorem 4. The spoke construction uses approximately $\ln n$ spokes $U_{p,\theta}(r)$, at evenly spaced angles. Almost always, for each spoke, every intersection of two consecutive ℓ_p-balls of radius $\lambda/2$ contains at least one vertex of V_n, provided that the bound (2) holds.

Let $v_1, v_2 \in V_n$. For $i = 1, 2$, by Corollary 1, there is a vertex $v_i' \in V_n$ lying inside some spoke U_i, connected to v_i by a path in $G_p(\lambda, n)$ of length $o(1/\lambda)$. Moreover, v_i' is connected to a vertex near the origin by a path consisting of vertices in $U_i \cap V_n$, lying in successive ℓ_p-balls of the spoke. Thus each of these two paths contains at most $\mathrm{diam}_p(D)/\lambda$ vertices, and concatenating these paths gives the desired upper bound on the diameter of $G_p(\lambda, n)$.

We can make the average Euclidean distance covered in a path from v_i' to v_j' larger by increasing r. This change decreases the area of intersection of consecutive ℓ_p-balls, which in turn requires an increase in c in order to guarantee a vertex of V_n in every region of intersection. This leads to the following corollary.

Corollary 2. *Let* $1 \leq p \leq \infty$ *and let* $\lambda = c\sqrt{\ln n/n}$. *For every* $\delta \in (0, 1]$, *there exists* $c_p(\delta) > 0$ *such that if* $c > c_p(\delta)$, *then* $G_p(\lambda, n)$ *is almost always connected, and has diameter at most* $\mathrm{diam}_p(D)(1 + \delta + o(1))/\lambda$ *as* $n \to \infty$.

References

1. Chen, X., Jia, X.: Package routing algorithms in mobile ad-hoc wireless networks. In: 2001 International Conference on Parallel Processing Workshops. (2001) 485–490
2. Stojmenovic, I., Seddigh, M., Zunic, J.: Dominating sets and neighbor elimination-based broadcasting algorithms in wireless networks. IEEE Trans. Parallel Distrib. Syst. **13** (2002) 14–25
3. Wu, J., Li, H.: A dominating-set-based routing scheme in ad hoc wireless networks. Telecommunication Systems **18** (2001) 13–36
4. Ellis, R.B., Jia, X., Yan, C.H.: On random points in the unit disk. (preprint)
5. Whittaker, E.T., Watson, G.N.: A course of modern analysis. Cambridge University Press (1996)
6. Penrose, M.D.: On k-connectivity for a geometric random graph. Random Structures Algorithms **15** (1999) 145–164
7. Alon, N., Spencer, J.H.: The probabilistic method. 2nd edn. John Wiley (2000)
8. Klarner, D.A.: Cell growth problems. Canad. J. Math. **19** (1967) 851–863