Network Probabilistic Connectivity: Using Node Cuts

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Abstract. The new algorithmic technique which allow higher efficiency in the precise calculation of the reliability of an undirected graph with absolutely reliable nodes and unreliable edges is considered in this paper. As graph's reliability we mean its probabilistic connectivity. The proposed technique is based on the reduction of the dimension of a calcuted graph by considering its node's cuts. Comprehensive computer simulation results show the advantages of the proposed algorithms, that the calculation time decreases significantly in comparison with existent methods.

1   Introduction

The task of calculating or estimating the probability of whether the network is connected (often referred to as its *reliability*, is the subject of much research due to its significance in a lot of applications, communication networks included. The problem is known to be NP-hard irrelative of whether the unreliable edges or nodes or both are considered. Most explored is the case of absolutely reliable nodes and unreliable edges that corresponds to real networks in which the reliability of nodes is much higher than that of edges. The transport and radio networks are good examples. Usually the estimations of a network reliability are considered. Yet we can conduct the exact calculation of reliability for networks with dimension of a practical interest by taking into consideration some special features of real network structures and based on modern high-speed computers. In [1] the technique was proposed that allows faster calculation based on reduction of simple chains and/or branching by them. This technique, in its turn, is based on the well-known factoring (branching) algorithm (often called Moore-Shannon algorithm [2]) uses branching on the alternative states for an arbitrary

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edge. Some good approaches are proposed in [3,4] but, as it is shown in [1], it is possible to improve them. In this paper we present new approach to reducing the task dimension by considering 2-node cuts. The proper theorems are proven in this paper to support the proposed methods.

The rest of the paper is organized as follows: in section 2 the basic notations and definitions are presented. In section 3 we prove the basic theorems while section 4 is devoted to computer algorithm and experimental results. Section 5 is the brief conclusion.

2   Basic Notations and Definitions

We denote an undirected graph as $G = (X, U)$, where X is a set of nodes and U – a set of edges of G. For each edge $u_i \in U$ the probability of its existence p_i is given. Further on we refer to this probability as *edge reliability*.

The *simple event* is an individual realization of a graph that is obtained by checking existence or non-existence of every edge.

The *probability of simple event* is obviously equal to

$$P(E) = \prod_{i \in Ex} p_i \prod_{i \notin Ex} (1 - p_i), \tag{1}$$

where Ex is the set of numbers of existent edges.

An arbitrary event (union of some simple events) we will consider as *successful* if all graphs that are correspondent to these simple events are connected that is that all nodes can be connected by existent edges.

Reliability of graph G, $R_P(G)$, is the probability of graph G being connected that is the probability of event that is the union of all successful events and of them only.

Let $\{x, y\}$ be an arbitrary 2-node cut in G which divides it into 2 subgraphs G_1 and G_2 as is shown in Fig. 1. Note that for any fixed cut $\{x, y\}$ G_1 and G_2 are not defined unambiguously in general case. For example, if there is an edge between x and y then it can belongs as to G_1 as to G_2.

We denote graphs that are obtained from G_1 and G_2 by contracting nodes x and y as G'_1 and G'_2, correspondingly. Let z be the node that is obtained by contracting nodes x and y.

3   Using Node Cuts for Reducing the Task Dimension

3.1   Reliability of Graph That Is Divided into 2 Components by 2-Node Cut

Here we prove the main theorem that correlates the reliability of a graph G with reliabilities of graphs G_1, G'_1, G_2, and G'_2.
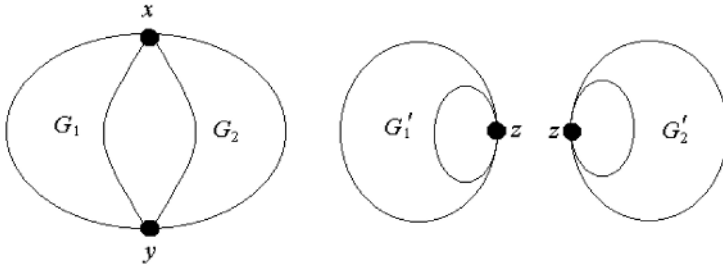
Fig. 1. Components of a graph G and auxiliary graphs

Theorem 1. *The following equation is true for any graph G that is divided into 2 components G_1 and G_2 by 2-node cut:*

$$R_P(G) = R_P(G_1) \cdot \big(R_P(G_2') - R_P(G_2)\big) + \tag{2}$$
$$R_P(G_2) \cdot \big(R_P(G_1') - R_P(G_1)\big) + R_P(G_1) \cdot R_P(G_2).$$

Proof. Let us denote:

A — union of all simple events that correspond to connected G_1 and G_2' and disconnected G_2.

B — union of all simple events that correspond to connected G_2 and G_1' and disconnected G_1.

C — union of all simple events that correspond to connected G_1 and G_2.

S — union of all simple events that correspond to connected G.

First we need prove the following statements:

1) Events A, B, and C are pairwise antithetical, that is $A \cap B = A \cap C = B \cap C = \emptyset$.
2) $A \cup B \cup C = S$.

The truth of the first statement is clear from the definitions of A, B, and C. Let us prove the second statement.

First we show the inclusion $A \subseteq S$. Let us consider an arbitrary simple event in A such that G_1 and G_2' are connected and G_2 is disconnected.

Let x_1 and x_2 be two arbitrary nodes of G_2. G_2' is connected so there is at least one path from x_1 to z and at least one path from x_2 to z in G_2'. Thus there are paths from x_1 to x or y and from x_2 to x or y in G_2. And if there exist paths from x_1 to x and from x_2 to x or from x_1 to y and from x_2 to y, then, obviously, x_1 and x_2 are connected by a path in G_2.

Now we will discuss the case when there are paths from x_1 and x_2 to different nodes (x or y). Without loss of generality we can assume that there are paths from x_1 to x and from x_2 to y. G_1 is connected so there exists a path from x to y in G_1 and, consequently, there exists a path from x_1 to x_2 through nodes x and y (see Fig.2 a)).
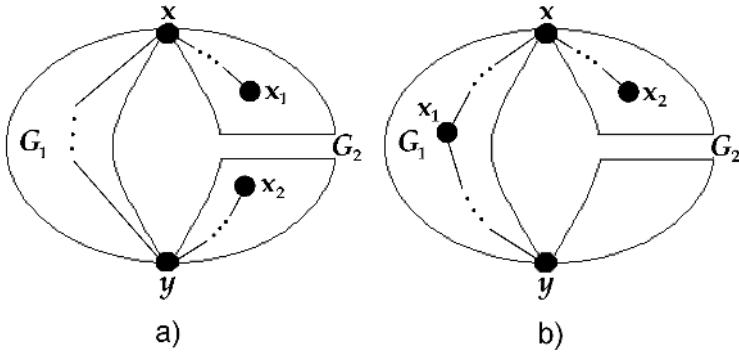
Fig. 2. Case of connected G_1 and G_2' and disconnected G_2

If x_1 is in G_1 and x_2 is in G_2 then x_1 is connected to both x and y and x_2 is connected to x or to y but not to both. Let ut assume that x_2 is connected to x for certainty. Thus x_1 and x_2 are connected by a path in G (see Fig.2 b)).

Thus any arbitrary pair of nodes in G are connected by at least one path in the case of event A and so $A \subseteq S$.

The inclusion $B \subseteq S$ is proved similar to $A \subseteq S$.

The inclusion $C \subseteq S$ is clear because a connected graph corresponds to any simple event in C. Thus $A \cup B \cup C \subseteq S$.

Now we transfer to proving the reverse inclusion, that is $A \cup B \cup C \supseteq S$.

Let us consider an arbitrary simple event E in S, that is some partial realization of G that is connected.

We need in three additional events:

D_1 — union of all simple events that corresponds to connected G_1 and disconnected G_2'.

D_2 — union of all simple events that corresponds to connected G_2 and disconnected G_1'.

D — union of all simple events that corresponds to disconnected G_1 and disconnected G_2.

It is clear that $A \cup B \cup C \cup D_1 \cup D_2 \cup D = \Omega$ where Ω is the complete event space. Note that events D_1, D_2 and D are obviously pairwise antithetical.

Let $E \notin A \cup B \cup C$. Then, according to the statement above, $E \in D_1 \cup D_2 \cup D$. Now we consider the cases $E \in D_1$, $E \in D_2$, and $E \in D$ separately.

1. $E \in D$. In this case G_1 and G_2 are disconnected so G is disconnected also. We have the contradiction because E is a connected realization of G.
2. $E \in D_1$. Let x_1 and x_2 be a pair of disconnected nodes in G_2' (such pair exists according to the definition of D_1). Specifically there is no path from x_1 to x_2 through z and this node corresponds to a path through connected G_1. Consequently they are disconnected in G. We have contradiction again.
3. $E \in D_2$. The contradiction is obtained similar to the case above.

Thus $E \notin D_1 \cup D_2 \cup D$ so $E \in A \cup B \cup C$ and, consequently, $S \subseteq A \cup B \cup C$. Then from $A \cup B \cup C \subseteq S$ we obtain $A \cup B \cup C = S$.

Now according to statements 1) and 2) we obtain $R(G) = P(S) = P(A) + P(B) + P(C)$.

The event C we can represent as intersection of two events which corresponds to connectivity of G_1 and connectivity of G_2, correspondingly. These events are obviously independent. Thus $P(C) = R(G_1) \cdot R(G_2)$.

The event $A = A_1 \cap A_2$, where A_1 corresponds to connectivity of G_1 and A_2 corresponds to connectivity of G'_2 and non-connectivity of G_2. These events are also independent so $P(A) = P(A_1) \cdot P(A_2)$.

Ii is clear that $P(A_1) = R(G_1)$. For obtaining $P(A_2)$ we present A_2 as intersection of events A_{21} and A_{22}, where A_{21} corresponds to non-connectivity of G_2 and A_{22} corresponds to connectivity of G'_2. Events A_{21} and A_{22} are dependent ones so $P(A_2) \neq P(A_{21}) \cdot P(A_{22})$. Let us derive the probability of A_2 in the following way.

By definition of A_{21} the event $\overline{A}_{21}$ corresponds to connectivity of G_2 thus $\overline{A}_{21} \subseteq A_{22}$ and $P(\overline{A}_{21}) = R(G_2)$. We have the obvious equivalence: $A_{22} \cap A_{21} = A_{22} \backslash \overline{A}_{21}$. From this we obtain

$$P(A_2) = P(A_{21} \cap A_{22}) = P(A_{22} \backslash \overline{A}_{21}). \tag{3}$$

From $\overline{A}_{21} \subseteq A_{22}$, we have

$$P(A_2) = P(A_{22} \backslash \overline{A}_{21}) = P(A_{22}) - P(\overline{A}_{21}) = R(G'_2) - R(G_2). \tag{4}$$

So

$$P(A) = P(A_1) \cdot P(A_2) = R(G_1) \cdot (R(G'_2) - R(G_2)). \tag{5}$$

In similar way we obtain that

$$P(B) = R(G_2) \cdot (R(G'_1) - R(G_1)). \tag{6}$$

As a result we have

$$R(G) = P(A) + P(B) + P(C) = R(G_1) \cdot (R(G'_2) - R(G_2)) + \tag{7}$$
$$R(G_2) \cdot (R(G'_1) - R(G_1)) + R(G_1) \cdot R(G_2),$$

just as expected. □

3.2 Reliability of a Graph That Is Divided into k Components by 2-Node Cut

Now let us consider the case when 2-node cut divides graph G into k subgraphs $G_1, G_2, \ldots, G_k$, $k > 2$.

First we need define a intermediate function S_k^i. Let us have $2k$ numbers $p_1, \ldots, p_k, p'_1, \ldots, p'_k$. Then S_k^i is equal to a sum of all possible productions of these numbers that satisfy the following restrictions:

1) each item has k co-factors;
2) each item has no co-factors with equal indexes;
3) each item has exactly i co-factors with stroke.

Thus

$$S_k^0 = p_1 p_2 p_3 \ldots p_k, \tag{8}$$
$$S_k^1 = p_1' p_2 p_3 \ldots p_k + p_1 p_2' p_3 \ldots p_k + \ldots + p_1 p_2 p_3 \ldots p_k',$$
$$S_k^2 = p_1' p_2' p_3 \ldots p_k + p_1' p_2 p_3' \ldots p_k + \ldots + p_1 p_2 p_3 \ldots p_{k-1}' p_k',$$
$$\ldots$$
$$S_k^{k-1} = p_1 p_2' p_3' \ldots p_k' + p_1' p_2 p_3' \ldots p_k' + \ldots + p_1' p_2' p_3' \ldots p_{k-1}' p_k.$$

As p_i and p_i' we consider reliabilities of G_i and G_i' where G_i' is obtained from G_i by contracting nodes x and y.

Theorem 2. *The following equation is true:*

$$R(G) = \sum_{i=0}^{k-1} (-1)^{k-i+1} S_k^i \tag{9}$$

Proof. We prove the theorem by induction.

1. For $k = 2$ we obtain the equation (2) that is proved to be true in previous consideration.

2. Let (9) be true for some k. Let us prove it for $k + 1$.

Let p_0 be the reliability of $G_1 \cup \ldots \cup G_k$ and p_{k+1} be the reliability of G_{k+1}. From (2) we have

$$R(G) = p_{k+1}(p_0' - p_0) + p_0(p_{k+1}' - p_{k+1}) + p_0 p_{k+1} = \tag{10}$$
$$p_{k+1} p_0' + p_{k+1}' p_0 - p_0 p_{k+1}.$$

But, by the assumption of induction

$$p_0 = \sum_{i=0}^{k-1} (-1)^{k-i+1} S_k^i, \tag{11}$$

and, obviously, $p_0' = p_1' \ldots p_k'$. Thus

$$R(G) = p_1' \ldots p_k' p_{k+1} + p_0(p_{k+1}' - p_{k+1}). \tag{12}$$

By simple substitution we can obtain the following property of S_k^i:

$$S_{k+1}^i = S_k^{i-1} p_{k+1}' + S_k^i p_{k+1}, \qquad 0 < i < k. \tag{13}$$

For $i = 0$ we have $S_{k+1}^0 = S_k^0 p_{k+1}$ and for $i = 1$ $S_{k+1}^k = S_k^{k-1} p_{k+1}' + p_1' p_2' \ldots p_k' p_{k+1}$.

Thus for odd k we have

$$\sum_{i=0}^{k}(-1)^{k-i+2}S_{k+1}^i = -S_{k+1}^0 + S_{k+1}^1 - \ldots - S_{k+1}^{k-1} + S_{k+1}^k =$$

$$-S_k^0 p_{k+1} + S_k^0 p'_{k+1} + S_k^1 p + \ldots - S_k^{k-1}p_{k+1} + S_k^{k-1}p'_{k+1} + p'_1 p'_2 \ldots p'_k p_{k+1} =$$

$$(p'_{k+1} - p_{k+1})(S_k^0 - S_k^1 + S_k^2 - \ldots + S_k^{k-1}) + p'_1 p'_2 \ldots p'_k p_{k+1} =$$

$$(p'_{k+1} - p_{k+1})\sum_{i=0}^{k-1}(-1)^{k-i+1}S_k^i + p'_1 p'_2 \ldots p'_k p_{k+1} =$$

$$(p'_{k+1} - p_{k+1})p_0 + p'_1 p'_2 \ldots p'_k p_{k+1} = R(G),$$

just as expected. Case of even k is proved in similar way.   □

4   Computational Algorithm and Case Studies

Based on the presented results we propose the modification of the algorithm for strict calculation of the graph reliability that is discussed in [1]. Briefly this modification looks as follows (we assume that a graph under consideration has connected structure).

Step 0 (preliminary). Remove all attached trees (if exist) from a graph, ω is the production of reliabilities of all removed edges.

Step 1. Substitute all simple chains (if exist) by edges according to formulas from [1], ν is the corresponding factor (if there are no simple chains in a graph then $\nu = 1$).

Step 2. Decompose a graph into k blocks (if decomposition is impossible, then $k=1$ and the only block is a graph itself) G_i. $R(G) = \nu \prod_{i=1}^{k} R(G_i)$. For each G_i go to next step.(if possible)

Step 3. If graph is disconnected or allows direct computation then return 0 or calculated value, correspondingly. Otherwise go to next step.

Step 4. If there is a 2-node cut in a graph then use equations (2) or (9) for recursion. Otherwise go to next step.

Step 5. Choose an arbitrary edge u with reliability ε for recursive branching: $R(G) = \nu[\varepsilon R(G^*) + (1 - \varepsilon)R(G \backslash e)]$. Here G^* is a graph G contracted by an edge u. Recursion starts from the Step 1.

Step 6 (final). Calculated value is multiplied by ω.

For case studies we use random graphs obtained according the algorithms from [5]. We have calculated the reliability of 100 random networks which structures that include 5 subgraphs corresponding to LANs and 5 additional edges for their interconnection. Each subgraph has 40 nodes and 60 edges thus each testing graph has 200 nodes and 305 edges. For experiments we have used the computer with the processor Intel Celeron 1000 MHz, RAM=320Mb. Average time for a random graph is about 80 sec while the previous algorithm that in [1] was

proven to be fastest at that time gives no results in an hour. At the same time the proposed algorithm is slightly slower for complete graphs (for K_9 we have 47 sec and 47.6 sec correspondingly) as a result of unfruitful tests on the Step 3. Thus our new algorithm has higher efficiency for graphs with 2-node cuts initially or when a graph has low density and do 2-node cuts are obtained often enough during the branching process.

5   Conclusion

In this paper we have shown how to use 2-node cuts for speeding up the process of obtaining the exact reliability of networks with reliable nodes and unreliable edges. Together with previous technique based on use of simple chains it allows calculation of reliability of LANs in reasonable time. We think that our method can be used for topological optimization of networks using the method proposed in [6] in which the calculation of a graph reliability is one of the main subgoals. In the nearest future we will present results concerning case of multiple 2-node cuts and of cuts of higher capacity. Future researches can concern the exact calculation of reliability for networks with unreliable nodes also.

Acknowledgment

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